# 11. AREAS BETWEEN CURVES

§11.1 Areas Between Curves

So areas above the *x*-axis are positive and areas below are negative, right? Wrong! I lied! Well, when you first learn about integration it's a convenient fiction that's true in a sense. But now we'll level with you and tell you the whole truth.

The boundaries for an area that's given by an integral are the graph of a function, two vertical lines and, up to now the fourth boundary has been the *x*-axis. We're now going to generalise that so that the fourth boundary is the graph of a second function. You can still take the second function to be $y = 0$ and that gives the *x*-axis. You can still have the *x*-axis as a boundary if you like. But you can, instead, have a curved bottom boundary.

We're going to develop a formula for the area between any two curves. There will be a 'top *y*' and a 'bottom *y*'. 'Top' and 'bottom' refer to which has the greater *y* values. Of course curves may cross over and this makes the problem just a little bit more complicated, but not much.

$y = $ f(x) (top y)

$y = $ g(x) (bottom y)

a   *b*
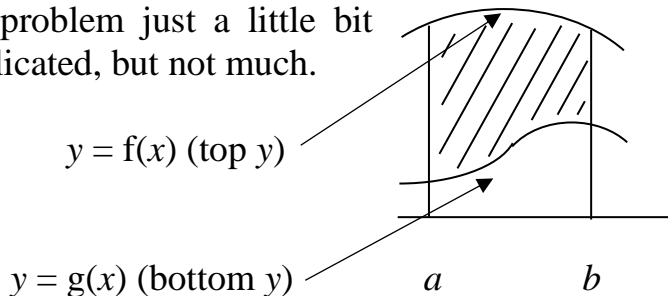
The shaded area is the area we want. Now $\int_a^b f(x)\,dx$ gives the area under the top curve, right down to the x-axis and $\int_a^b g(x)\,dx$ gives the area under the bottom curve, right down to the x-axis.

Clearly the area we want is the difference between the two, that is, $\int_a^b f(x)\,dx - \int_a^b g(x)\,dx$

But this can be written as $\int_a^b [f(x) - g(x)]\,dx$.

If $f(x) \geq g(x)$ for $a \leq x \leq b$,
the area between $y = f(x)$ and $y = g(x)$
between $x = a$ and $x = b$ is

$$\int_a^b [f(x) - g(x)]\,dx$$

To remember this, just remember the integral as:

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top limit
$$\int (\text{top } y - \text{bottom } y)$$
bottom limit

Now when we have just one curve $y = f(x)$ which lies above the x-axis from $x = a$ to $x = b$ and the bottom

boundary of the area is the *x*-axis, then the top y is $y = f(x)$ and the bottom *y* is $y = 0$. So the area under the graph is

$$\int_a^b [f(x) - 0] \, dx = \int_a^b f(x) \, dx \text{, as before.}$$

But if $y = f(x)$ lies below the *x*-axis this becomes the bottom boundary and the *x*-axis, $y = 0$ becomes the top boundary.

The area becomes, in this case, $\int_a^b [0 - f(x)] \, dx$

$$= - \int_a^b f(x) \, dx \, .$$

Previously we explained this minus sign by saying that "areas below the *x*-axis are negative". Now we're in a position to set the record straight. Areas are never negative, but integrals may be. Where the curve is below the *x*-axis the area is positive but the integral is negative. That's the right way to look at it.

  Whenever you have to work out areas below, above or between curves always set it up as a problem involving the area between two curves. If you correctly identify which is the 'top *y*' and which is the 'bottom *y*' you'll never go wrong. The integral you set up will always come out as positive and will give the required area. Integrals come out negative, when dealing with areas, only if you haven't set things up properly.

  There's a lot of sloppy thinking with this topic and students are often taught bad habits. Alright we did start

you off thinking that "areas below the axis are negative" but at the time it would have confused you to do otherwise. At least we're now setting the story right.

Some students think that since areas are positive but integrals can be negative, all you have to do is to put absolute value signs around everything. Even worse is the student who, on evaluating an integral to be –2, writes "= – 2 = 2". Using absolute value signs display ignorance as to what is really going on and can lead to wrong answers if the curves cross over.
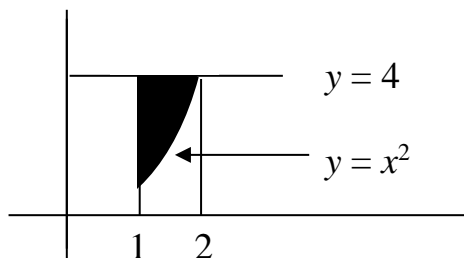
So some guidelines to avoid such errors are:

Never 'fudge' the sign if the integral comes out negative.
Never use absolute value signs in connection with areas.

Example 1:
Find the area between $y = x^2$, $y = 4$ and $x = 1$.

**Solution:** Top y is $y = 4$ and bottom y is $y = x^2$. The area is therefore $\int_{1}^{2}[4 - x^2]\,dx = \left[4x - (1/3)x^3\right]_{1}^{2}$

$$= \left(8 - \frac{8}{3}\right) - \left(4 - \frac{1}{3}\right) = \frac{5}{3}.$$

§11.2 What If Curves Cross?

It's absolutely important to determine whether curves cross within the region being considered. For if they do, what was the 'top y' will become the 'bottom y' and vice versa. The interval of integration will have to be split up.

Suppose we want the area between the curves $y = f(x)$ and $g(x)$ between $x = a$ and $x = b$ and suppose the curves cross at some point c between a and b. Suppose that $g(x) \geq f(x)$ for $x \leq c$ and $g(x) \leq f(x)$ for $x \geq c$. Then from $x = a$ to $x = c$ it's $g(x)$ which is the top y, while from $x = c$ to $x = b$ it's $f(x)$ that's on top.
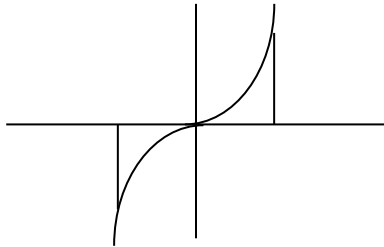
We must break the interval $[a, b]$ into two pieces and integrate separately over these two regions.
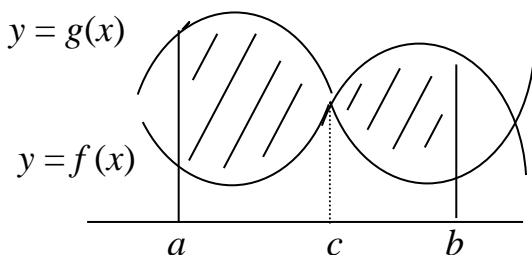
The required area will be:

$$\int_a^c [g(x) - f(x)]\, dx \;+\; \int_c^b [f(x) - g(x)]\, dx\,.$$

Example 2: Find the area between $y = x^3$, the x-axis and lines $x = -1$ and $x = 1$.

Solution: Because the curve cuts the x-axis at $x = 0$ we must divide the interval $[-1, 1]$ into two pieces. From $x = -1$ to $x = 0$ the top y is $y = 0$ (the x-axis) while from $x = 0$ to $x = 1$ it is $y = x^3$.

The required area is $\displaystyle\int_{-1}^{0} [0 - x^3]\, dx + \int_{0}^{1} [x^3 - 0]\, dx$

$$= \left[-\frac{x^4}{4} \right]_{-1}^{0} + \left[\frac{x^4}{4} \right]_{0}^{1} = \left(0 - \left(-\tfrac{1}{4} \right) \right) + \left(\tfrac{1}{4} - 0 \right) = \tfrac{1}{2}\,.$$

Two things should be noted with this example.
(1) If we had simply integrated x^3 between -1 and 1 we would have concluded that the area is:

$$\int_{-1}^{1} x^3 \, dx = \left[\frac{x^4}{4}\right]_{-1}^{1} = \frac{1}{4} - \frac{1}{4} = 0,$$ which is clearly wrong. And taking absolute values wouldn't help either. Remember to *never* use absolute values with these area questions. If you don't know which is top y and which is bottom you're likely to come up with a totally wrong answer. If you do, then you'll set up the integral correctly so that it will come out positive.

(2) We could have exploited symmetry to simplify the calculation and avoid all those minus signs. The area below the x-axis is clearly the same as the area above so we could have said that the total area is

$$2\int_{0}^{1} x^3 \, dx = 2\left[\frac{x^4}{4}\right]_{0}^{1} = \frac{1}{2}.$$

Example 3: Find the area between $y = x^2$ and $y = x$.
Solution: Here we're not given any endpoints. That's because the graphs of $y = x^2$ and $y = x$ cut in exactly two places and so they enclose a region. We need to find these two points.
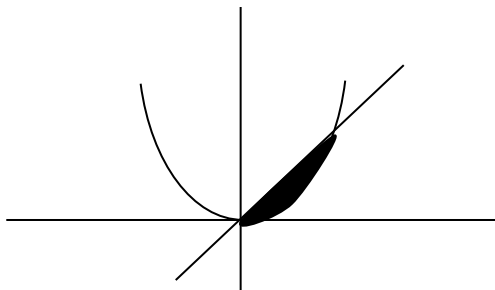
Solving $x^2 = x$ we get $x = 0$ or 1. So these are the limits of integration. The top y, between 0 and 1, is $y = x$. The area is therefore $\int_0^1 [x - x^2]\, dx$. Because we know what we're doing and have set up the integral correctly we can be sure that it will be positive.
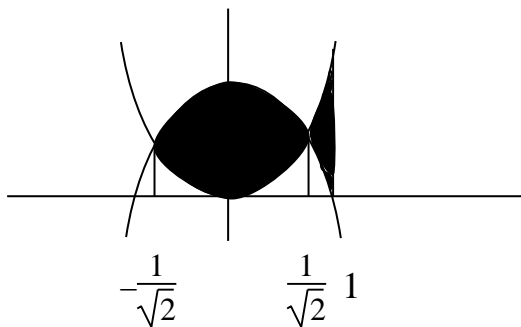
$\int_0^1 [x - x^2]\, dx = \left[\dfrac{x^2}{2} - \dfrac{x^3}{3}\right]_0^1 = \dfrac{1}{2} - \dfrac{1}{3} = \dfrac{1}{6}$. This is the required area.

Example 4: Find the area between $y = x^2$, $y = 1 - x^2$ and $x = 1$.

Solution: The curves cut when $x^2 = 1 - x^2$, that is when $2x^2 = 1$. So they cut in two places, at $x = \pm\dfrac{1}{\sqrt{2}}$.

We have to integrate from $-\dfrac{1}{\sqrt{2}}$ to $\dfrac{1}{\sqrt{2}}$ and then from $\dfrac{1}{\sqrt{2}}$ to 1. By symmetry the integral from

$-\dfrac{1}{\sqrt{2}}$ to $\dfrac{1}{\sqrt{2}}$ is double that from 0 to $\dfrac{1}{\sqrt{2}}$.

The area is therefore:

$$2 \int_{0}^{1/\sqrt{2}}[(1-x^2)-x^2]\,dx + \int_{1/\sqrt{2}}^{1}[x^2-(1-x^2)]\,dx$$

$$= 2 \int_{0}^{1/\sqrt{2}}[1-2x^2]\,dx + \int_{1/\sqrt{2}}^{1}[2x^2-1]\,dx$$

$$= 2\left[x-\frac{2}{3}x^3\right]_{0}^{1/\sqrt{2}} + \left[\frac{2}{3}x^3-x\right]_{1/\sqrt{2}}^{1}$$

$$= 2\left(\frac{1}{\sqrt{2}}-\frac{1}{3\sqrt{2}}\right)+\left(\frac{2}{3}-1\right)-\left(\frac{1}{3\sqrt{2}}-\frac{1}{\sqrt{2}}\right)$$

$$= \left(\frac{1}{3\sqrt{2}}\right)(6-2-1+3)-\frac{1}{3} = \frac{6}{3\sqrt{2}}-\frac{1}{3}$$

$$= \frac{2}{\sqrt{2}}-\frac{1}{3} = \sqrt{2}-\frac{1}{3}.$$

EXERCISES FOR CHAPTER 11

Exercise 1: Find the area between the *y*-axis and the parabolas $y = x^2$ and $y = 1 - x^2$. Give your answer as an exact value, in terms of $\sqrt{2}$, as well as an approximation to 4 decimal places.

Exercise 2: Find the area enclosed between the curves $y = x^2 - 2x$ and $x^3 - 2x^2$.

Exercise 3: Find the area enclosed between the curves $y = e^x$ and $y = 6x - x^2 - 20$ between $x = 0$ and $x = 3$. Give your answer as an exact value involving e^3 as well as an approximate numerical value to 4 decimal places.

Exercise 4: Find the area enclosed between the curve $y = \dfrac{6}{x}$ and the line $y = 5 - x$. Give the answer to 4 decimal places without using your calculator, but instead using the fact that, to 4 decimal places, $\log 1.5 = 0.4055$.

Exercise 5: Use integration to find the area between the lines $y = 1 + 2x$ and $y = 1 - x$ between $x = -5$ and $x = 1$. Check your answer using simple geometry.

Exercise 6: Find the area enclosed between the curves $y = \sqrt{4x + 5}$, $y = \sqrt{5x + 4}$ and the x-axis.

Exercise 7: Find the area between $y = \sin x$ and $y = \cos x$ between two successive points where the curves intersect.

Exercise 7: This area $= \displaystyle\int_{\pi/4}^{7\pi/4} [\sin x - \cos x]\, dx$

$$= \left[\sin x - \cos x\right]_{\pi/4}^{7\pi/4}$$

$$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = \frac{4}{\sqrt{2}} = 2\sqrt{2}.$$

Exercise 7: In Economics, the Lorenz curve is a graph that's used in economics that shows the distribution of income. If the bottom $100x$ % of households represents the bottom $100y$ % of income, the point (x, y) lies on the Lorenz curve. The line $y = $ x corresponds to perfect equality, but in practice the curve lies below it. The most extreme case, (perfect inequality) would be where one household earns all of the income and the rest earn nothing. This corresponds to the line $y = 0$ (with a spike at $x = 1$). The Gini coefficient is the ratio of the area between the line of perfect equality and the observed Lorenz curve, and the area between the line of perfect equality and the line pf perfect inequality. It is a measure of inequality. The higher the Gini coefficient, the more unequal is the distribution of income.

Find the Gini coefficient if the Lorenz curve is $y = x^2$.

line of perfect equality

observed Lorenz curve

line of perfect inequality
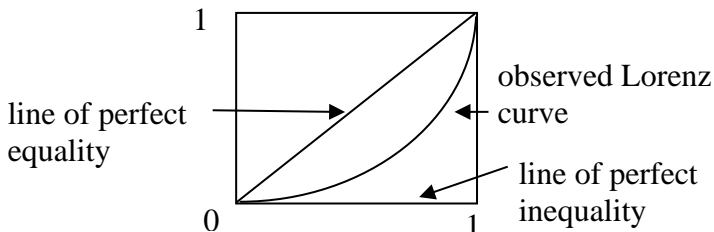
SOLUTIONS FOR CHAPTER 11

Exercise 1: The curves cut when $x^2 = 1 - x^2$, that is, when $2x^2 = 1$ or $x = \dfrac{1}{\sqrt{2}}$.
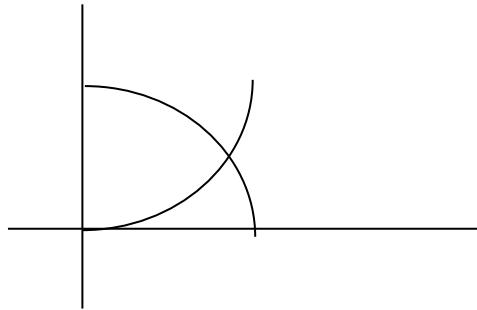
Over the range from $x = 0$ to $x = \dfrac{1}{\sqrt{2}}$ the top y is $y = 1 - x^2$.

So the area is $\displaystyle\int_{0}^{1/\sqrt{2}} (1 - x^2) - x^2 \, dx = \int_{0}^{1/\sqrt{2}} 1 - 2x^2 \, dx =$

$\left[x - x^2 \right]_{0}^{1/\sqrt{2}} = \dfrac{1}{\sqrt{2}} - \dfrac{1}{2} \approx 0.2071.$

Exercise 2: The curves cross if $x^2 - 2x = x^3 - 2x^2$. Solving, we get $x^3 - 3x^2 + 2x = 0$. Factorising we get $x(x - 1)(x - 2) = 0$, so the curves cross when $x = 0$, 1 and 2. The area enclosed is the region between the curves from $x = 0$ to x

= 2. But since they cross at $x = 1$ we need to split the interval $[0, 2]$ into two pieces.

From $x = 0$ to $x = 1$ the top curve is $y = x^3 - 2x^2$.
Then from $x = 1$ to $x = 2$ the top curve is $y = x^2 - 2x$.

So the area is: $A = \int_0^1 [(x^3 - 2x^2) - (x^2 - 2x)]\, dx +$

$\int_1^2 [(x^2 - 2x) - (x^3 - 2x^2)]\, dx$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_0^1 + \left[x^3 - \frac{x^4}{4} - x^2\right]_1^2$$

$$= \left(\frac{1}{4} - 1 + 1\right) - 0 + \left(8 - \frac{16}{4} - 4\right) - \left(1 - \frac{1}{4} - 1\right) = \frac{1}{2}.$$

Exercise 3: The curves do not cross and the top y is $y = e^x$.

So the area is $\int_0^3 [e^x - (6x - x^2 - 20)]\, dx$

$$= \int_0^3 [e^x - 6x + x^2 + 20]\, dx)$$

$$= \left[e^x - 3x^2 + \frac{1}{3}x^3 + 20x\right]_0^3$$

$$= (e^3 - 27 + 9 + 60) - 1$$

$$= e^3 + 41 \approx 61.0855.$$

Exercise 4: The curves cut when $\dfrac{6}{x} = 5 - x$. Solving we
get $6 = 5x - x^2$ and so $x^2 - 5x + 6 = 0$.
Hence $(x - 2)(x - 3) = 0$ and so $x = 2, 3$.
Let's see what happens between $x = 2$ and $x = 3$.

When $x = 2.5$ the value of y for the curve $y = \dfrac{6}{x}$ is 2.4
while for the line $y = 5 - x$ it is 2.5.
So between $x = 2$ and $x = 3$ the top y is $y = 5 - x$.
The area between is therefore:

$$\int_{2}^{3} [(5 - x) - \frac{1}{x}] \, dx \; = \; [5x - x^2 - \log x]_{2}^{3}$$

$$= (15 - 9 - \log 3) - (9 - 4 - \log 2)$$
$$= 1 - \log 3 + \log 2$$
$$= 1 - \log(3/2)$$
$$\approx 0.5945.$$

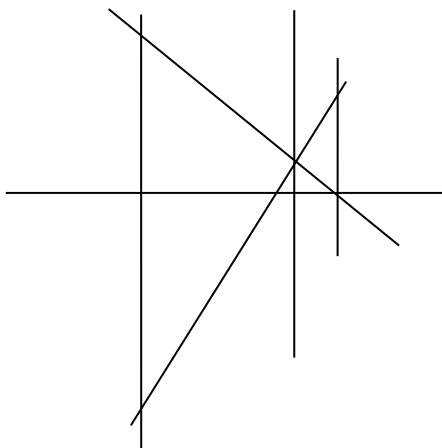**Exercise 5:** The lines cut at $x = 0$.
When $x = -1$, the value of y for $y = 1 + 2x$ is -1 while for
$y = 1 - x$ it is 2, so between $x = -5$ and $x = 0$ the top y is y
$= 1 - x$. Between $x = 0$ and $x = 1$ the top y is $y = 2x + 1$.
So the enclosed area is:

$$\int_{-5}^{0}[(1 - x) - (2x + 1)] \, dx \; + \int_{0}^{1}[(2x + 1) - (1 - x)] \, dx$$

$$= \int_{-5}^{0}[-3x]\,\mathrm{d}x \ + \int_{0}^{1}[3x]\,\mathrm{d}x \ = \left[-\frac{3}{2}x^2\right]_{-5}^{0} \ + \left[\frac{3}{2}x^2\right]_{0}^{1}$$

$$= (0 + \frac{3}{2}.25) + (\frac{3}{2} - 0)$$

$$= \frac{75}{2} + \frac{3}{2}$$

$$= \frac{78}{2}$$

$$= 39.$$


We can check this by simple geometry since the region consists of two triangles. Remember that the area of a triangle is ½ × base × perpendicular height. Here we take the 'base' to be the vertical sides.

When $x = -5$, $1 - x = 6$ while $2x + 1 = -9$. So the larger triangle has a base of 15 and a perpendicular height of 5. Its area is $\frac{15 \times 5}{2} = \frac{75}{2}$.

When $x = 1$, $2x + 1 = 3$ and $1 - x = 0$ so the smaller triangle has base 3 and perpendicular height of 1. Its area is $\frac{3}{2}$. The total area is therefore $\frac{75}{2} + \frac{3}{2} = 39$.

Exercise 6: The curves cross when $4x + 5 = 5x + 4$, that is, when $x = 1$.
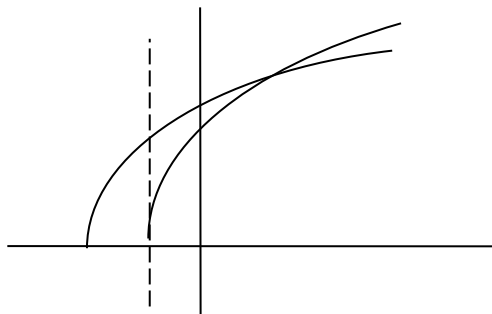
The curve $y = \sqrt{5x + 4}$ cuts the x-axis at $x = -4/5$ while $y = \sqrt{4x + 5}$ cuts it at $x = -5/4$.
We must divide the area into two portions at $x = -4/5$. he top y between $x = -5/4$ and $x = 1$ is $\sqrt{4x + 5}$.

Between $x = -5/4$ and $-4/5$ the bottom y is $y = 0$, the x-axis. Then between $x = -4/5$ and $x = 1$ the bottom y is $y = \sqrt{5x + 4}$.

Hence the area is:

$$\int_{-5/4}^{-4/5}\sqrt{4x + 5}\ \mathrm{d}x \ + \ \int_{-4/5}^{1}[\sqrt{4x + 5} - \sqrt{5x + 4}]\ \mathrm{d}x$$

$$= \left[\frac{2}{3} \cdot \frac{1}{4}(4x+5)^{3/2} \right]_{-5/4}^{-4/5}$$

$$+ \left[\frac{2}{3} \cdot \frac{1}{4}(4x+5)^{3/2} - \frac{3}{2} \cdot \frac{1}{5}(5x+4)^{3/2} \right]_{-4/5}^{1}$$

$$= \frac{1}{6} \left[\left(-\frac{16}{5} + \frac{25}{5} \right)^{3/2} - 0 \right] + \frac{1}{6} \left[9^{3/2} - \left(-\frac{16}{5} + \frac{25}{5} \right)^{3/2} \right]$$

$$- \frac{3}{10} [9^{3/2} - 0]$$

$$= \frac{1}{6} \left[\left(\frac{9}{5} \right)^{3/2} \right] + \frac{1}{6} \left[9^{3/2} - \left(\frac{9}{5} \right)^{3/2} \right] - \frac{3}{10} 9^{3/2}$$

$$= \frac{1}{6} \left[\frac{27}{5\sqrt{5}} \right] + \frac{1}{6} \left[27 - \frac{27}{5\sqrt{5}} \right] - \frac{3}{10} \cdot 27$$

$$= \frac{9}{2} - \frac{81}{10} = 0.45.$$

Exercise 7: The area between the line of perfect equality $(y = x)$ and the observed Lorenz curve, is

$$\int_0^1 (x - x^2 \, dx) = \left[\frac{1}{2} x^2 - \frac{1}{3} x^3 \right]_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}.$$

The area between the line of perfect equality $(y = x)$ and the line of perfect inequality $(y = 0)$ is the area of the triangle below the line $y = x$, which is clearly ½ . Hence the Gini coefficient is $\frac{1/6}{1/2} = \frac{1}{3}$.