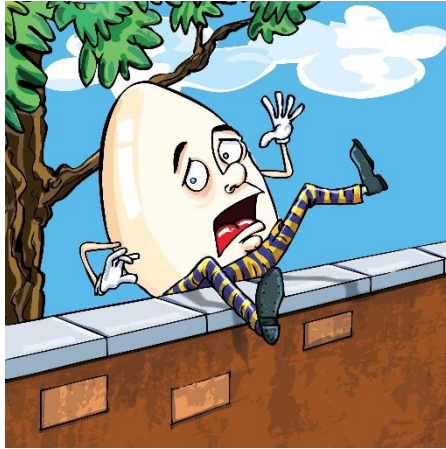


1. LOGIC

*There existed an egg who sat on a wall,
And the wall being short implies this story is tall.
Now if that fat egg had had a great fall
Or slipped off the top, but not jumped, then not all
The king's horses and all the king's men,
If they worked through the day and the evening, then
They could not succeed if and only if when
They attempted to put Humpty together again.*



§1.1. The Role of Logic in Mathematics

Over many hundreds of years mathematics has played a vital role in the development of the sciences and so it is often regarded as a science itself. But mathematics is quite different to any of the other sciences in one important respect. It has no empirical facts. Physicists carry out experiments and formulate hypotheses. They

use mathematics as a language to describe their hypotheses and as a tool to explore their logical consequences. But in the final analysis physical theories must be supported by experimental facts.

Where mathematics appears to come closest to being an experimental science is in geometry. We measure the angles of a triangle and they appear to total 180° . But when we do that we're acting as experimental physicists. We now know that there's no logical necessity for the sum of the angles of a triangle to be 180° . While this *is* a fact in Euclidean geometry, the geometry we all learnt at school, there are other logically consistent geometries where this is not the case. Could it not be that the angles of a triangle add up to slightly more than 180° or a little less. Perhaps the discrepancy is so minute that it hasn't yet been detected in our measurements. Who's to say whether Euclidean geometry is the right one to describe physical space? Only a physicist could do this with careful measurement – certainly not a mathematician.

In a very real sense, mathematics has no facts. It's concerned with the logical relationships *between* facts. Well then, is mathematics just the same as logic? No. Logic provides the tools for reasoning. Mathematics *uses* these logical tools to construct elaborate systems which can be used by science as models for different aspects of the real world.

If mathematics was to be viewed as a science, and there are those who argue that it should be, then its

laboratory is the human mind. Its instruments are the techniques of proof. This is why it is important for anyone studying mathematics to have an appreciation of logic.

§1.2. The Role of Logic in Computing

What was said about logic in mathematics is also true of computing science because in theoretical computer science there are theorems and proofs just as in mathematics. In fact there are those chauvinist mathematicians who dismiss theoretical computer science and theoretical physics as just particular branches of mathematics!

In one sense they're right. But computer science and theoretical physics are sufficiently integrated with experimental and practical aspects that, for practical purposes, they are quite properly studied in their own specialised departments. However you should never forget that the boundary between theoretical computing science and discrete mathematics is somewhat arbitrary. There is really a continuum that takes us all the way from mathematics to computer science and back again.

So, just as logic is important to mathematics, it's important to computer science. But for computer scientists, logic plays another very important role which affects some of the most applied areas of that subject.

It's no accident that at the heart of the central processing chip there's the 'arithmetic logic unit'. We talk about the logic of computer programs, for very good reasons. At a higher level still, to understand the theory of

databases one needs to understand logic, and its partner, set theory. And another area, of interest to both mathematicians and computing scientists alike, is the area of automated theorem proving and proofs of program correctness.

§1.3. Propositions and Truth Functions

Definition: A **proposition** is a statement for which it is meaningful to say that it is true or false (but not both).

The **truth value** of a proposition p is T or F according as the proposition is false or true.

$$\mathbf{T = TRUE \quad F = FALSE}$$

Now we don't need to ask the deep question "what is truth?" For our purpose truth values can be thought of as tags that are given to certain statements. Logic is concerned with *relative* truth, that is, the truth of a compound statement given the truth of its primitive constituents and the truth of those primitive statements is established by other means, or is simply assumed in axioms or definitions.

Example 1

"It rained today" is a proposition while "Will it rain today?" or "Come inside, it's raining." are not. Some statements appear to be propositions, but on closer examination they are not because to attach a truth value to them leads to a logical paradox.

Example 2

THIS STATEMENT IS FALSE

is not a proposition because if we say that it is true then it's false and if we say that it is false then it's true.

It's the self-referential nature of this statement, the fact that it purports to be saying something about itself, which appears to cause the problem. However there are cases, involving no self-referentiality, that still lead to trouble.

Example 3

Consider the following infinite list of statements:

At least one of the statements below is FALSE.

At least one of the statements below is FALSE.

At least one of the statements below is FALSE.

At least one of the statements below is FALSE.

At least one of the statements below is FALSE.

.....

There is no self-referentiality here. Each statement purports to be saying something about all the remaining ones. So, although each reads identically to the next, they're all different statements, referring to a different set of statements, and so they don't all have to have the same truth value.

But if any one of them is FALSE then all the remaining ones are TRUE and it is easy to see that this leads to a contradiction. But if they're all true we again get a

contradiction. Think this out carefully. If it is FALSE that all the following statements are FALSE then one of the following statements must be TRUE!

Such problems can keep a logician awake at night, but we take the more practical view that we'll ignore such potential problems and hope that we never meet them. That's a reasonable practical attitude because such logical paradoxes only exist in the laboratory. One has to go out of one's way to produce them.

§1.4. Compound Propositions

Compound propositions can be made up from simpler ones in such a way that their truth value can be determined from those of their constituents. We use truth 'tables'. These set out the truth value of compound statements based on the truth values of their constituents. Note that you don't need to know what the statements are – just whether they are TRUE or FALSE.

For example, if p is TRUE and q is FALSE then “p and q” is FALSE while “p or q” is TRUE.

Not, And, Or

The simplest truth operator is 'not' which only involves one constituent. Its truth table is:

p	not p
T	F
F	T

The truth tables for ‘and’ and ‘or’ are:

p	q	p and q	p	q	p or q
T	T	T	T	T	T
T	F	F	T	F	T
F	F	F	F	T	T
F	T	F	F	F	F

Notice that ‘p or q’ is defined to be TRUE even when both are true. Sometimes in everyday English we use ‘or’ in an exclusive sense, but in logic, and in mathematics, it always includes the possibility of both.

Richard Feynman, a famous physicist, was once asked by the Dean’s wife at Princeton University whether he wanted milk or lemon in his tea. He replied “both”. “Surely you’re joking Mr Feynman.” He was, but he was making a point about the mathematician’s use of the word ‘or’ – it *can* include both.

In order to work at the level of the underlying logical structure we denote primitive propositions (ones which are not built up from simpler ones) by letters of the alphabet just as in algebra we represent numbers by letters.

The above three truth operators are denoted by special symbols \neg , \wedge , \vee respectively.

$\neg p$ denotes 'not p';
 $p \wedge q$ denotes 'p and q';
 $p \vee q$ denotes 'p or q';

Several variations are in common use:

'not p' is often denoted by p' , $\neg p$, or \bar{p} .
 Sometimes 'p and q' is denoted by $p \cdot q$ and
 'p or q' can be written as $p + q$.

Example 4

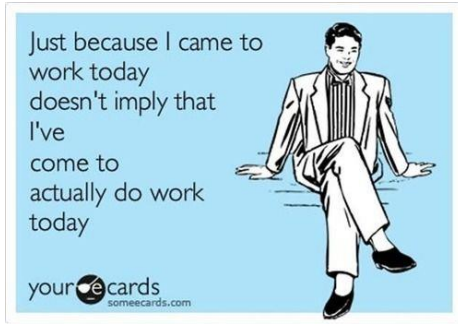
$(p \vee q) \wedge \neg(p \wedge q)$ denotes the compound statement 'p or q but not both'. This is the 'exclusive or'. Also note the use of the word 'but' here. It is simply an alternative to 'and'. While there may be overtones of contrast or emphasis with 'but' that we don't get with 'and', at the fundamental level of logic their meaning is identical.

Implication

Our next truth operator is one that's very much misunderstood – implication. The definition of 'p implies q' is given by its truth table:

p	q	p implies q
T	T	T
T	F	F
F	T	T
F	F	T

The problem that many people have with this definition is the third row which says that a false proposition implies a true one! In fact all it shows is that the technical definition of implication differs somewhat from the ordinary sense of the word.



In normal usage implication involves a causal connection. It might be the case that I'm wealthy and that I'm honest. In the ordinary sense of the word we would not say however that "being wealthy implies that I am honest". Wealth does not cause honesty. However if both propositions are true for me then, in the sense of propositional logic, "I am wealthy" implies "I am honest".

Because propositional logic deals with isolated propositions it can't express the notion of wealthy people *always* being honest (or its negation). That requires 'quantifiers', something we'll discuss later.

If you still feel uneasy about the above definition of implication you might like to ask yourself how else you might define it. For example if you decide that false statements can't imply anything then you'd want to change the last two rows of the truth table to become:

p	q	?
T	T	T
T	F	F
F	T	F
F	F	F

But this is just the truth table for ‘and’ and surely ‘implies’ does not mean the same as ‘and’. The final vindication for the accepted definition however only comes when we see implication in the context of quantifiers (see §1.8).

Notation: We denote ‘p implies q’ by $\mathbf{p \rightarrow q}$. Sometimes this is written as $p \Rightarrow q$ or $p \supset q$.

Equivalence

We say that p is (logically) **equivalent to** q if they have the same truth value.

We’ll denote ‘p is equivalent to q’ by $\mathbf{p \leftrightarrow q}$.

Other notations in use are $p \equiv q$ and $p \Leftrightarrow q$.

This definition can be set out in a truth table:

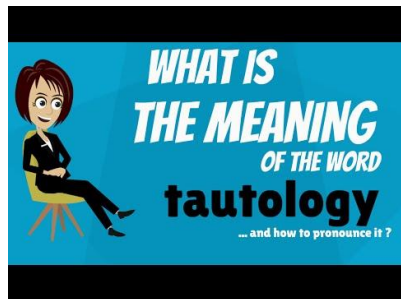
p	q	p ↔ q
T	T	T
T	F	F
F	T	F
F	F	T

Summary of Truth Operators

p	q	$\neg p$	$p \wedge q$	$p \vee q$	$p \rightarrow q$	$p \leftrightarrow q$
T	T	F	T	T	T	T
T	F	F	F	T	F	F
F	T	T	F	T	T	F
F	F	T	F	F	T	T

§1.5. Tautologies

A **tautology** is a proposition built up from primitive propositions, which is always true irrespective of the truth values of the constituent propositions.



Tautologies are logical theorems. For example ‘(p and q) implies (q and p)’ doesn’t give us any information about the statements p and q. Rather it tells us about the symmetry of the ‘and’ operator.

Example 5: The following three propositions are tautologies: (1) $p \leftrightarrow p$

$$(2) p \rightarrow (p \vee q)$$

$$(3) (p \wedge q) \leftrightarrow (q \wedge p)$$

But $(p \vee q) \rightarrow (p \wedge q)$ is not a tautology.

The above tautologies seem intuitively obvious. More complicated statements could be very difficult to test.

However any proposition can be tested mechanically to determine whether or not it is a tautology. We use a ‘truth table’, where all possibilities are considered.

Example 6: Is $\neg(\neg p \vee \neg q) \leftrightarrow p \wedge q$ a tautology?

We consider each of the four possible combinations of truth values (with a proposition involving p , q and r there are eight combinations) and work out the resulting truth value of the whole statement.

A very compact way of setting this out is in a table where each row corresponds to one such combination. We write the truth values for the primitive statements p and q underneath those symbols and the truth values for each sub-proposition underneath the symbol for the relevant operator.

$\neg(\neg p \vee \neg q) \leftrightarrow p \wedge q$									
T	F	T	F	F	T	T	T	T	T
F	F	T	T	T	F	T	T	F	F
F	T	F	T	F	T	T	F	F	T
F	T	F	T	T	F	T	F	F	F
9	5	1	8	6	2	10	3	7	4

The last row isn’t usually included. It’s given here simply to show the order in which the columns were filled in. This order can be varied a little but, for example, the first column can’t be filled in until the \vee column (representing the proposition to be negated) has been completed.

§1.6. Translating From English

$p \rightarrow q$ might be expressed as:

if p then q

p implies q

q is implied by p

q , if p

p only if q

p is a sufficient condition for q

q is a necessary condition for p

$p \leftrightarrow q$ might be expressed as:

p is equivalent to q

p if and only if q

(sometimes ‘if and only if’ is abbreviated to ‘**iff**’)

p is a necessary and sufficient condition for q

$p \wedge q$ might be expressed as:

p and q

not only p , but q

$p \vee q$ might be expressed as:

p or q

unless p then q

p , unless q

at least one of p and q

Miscellaneous constructions:

p but not q

neither p nor q

$p \wedge \neg q$

$\neg (p \vee q)$

p or q but not both $(p \vee q) \wedge \neg (p \wedge q)$
or more simply as $p \leftrightarrow \neg q$

[To see that these are equivalent use a truth table.]

§1.7. Laws of Logic

The following is a list of tautologies that are commonly used. Most of them are intuitively obvious but all of them can be tested using truth tables. Examine them, think about them, but *do not attempt to learn the list*.

To avoid complexity we use the convention that there are implied parentheses around each side of an implication or an equivalence and we assume that the negation operator only applies to the following proposition. If we wish to express something different then we need to include parentheses.

For example $p \vee q \leftrightarrow r$ means $(p \vee q) \leftrightarrow r$. If we wish to express $p \vee (q \leftrightarrow r)$ then the parentheses are needed. Also $\neg p \wedge q$ means $(\neg p) \wedge q$. If we wish to express $\neg (p \wedge q)$ we need the parentheses.

Commutative Laws:

$$(1) p \vee q \leftrightarrow q \vee p$$

$$(2) p \wedge q \leftrightarrow q \wedge p$$

Associative Laws:

$$(3) (p \vee q) \vee r \leftrightarrow p \vee (q \vee r)$$

$$(4) (p \wedge q) \wedge r \leftrightarrow p \wedge (q \wedge r)$$

Distributive Laws:

$$(5) p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)$$

$$(6) p \vee (q \wedge r) \leftrightarrow (p \vee q) \wedge (p \vee r)$$

Idempotent Laws:

$$(7) p \vee p \leftrightarrow p$$

$$(8) p \wedge p \leftrightarrow p$$

De Morgan Laws:

$$(9) \neg (p \vee q) \leftrightarrow \neg p \wedge \neg q$$

$$(10) \neg (p \wedge q) \leftrightarrow \neg p \vee \neg q$$

Miscellaneous Laws:

$$(11) p \wedge q \rightarrow p$$

$$(12) p \rightarrow p \vee q$$

$$(13) \neg(\neg p) \leftrightarrow p$$

$$(14) p \vee \neg p$$

$$(15) \neg(p \wedge \neg p)$$

These last two assert that a proposition must either be true or false but not both.

Syllogism:

$$(16) (p \rightarrow q) \wedge p \rightarrow q$$

A **syllogism** is a logical argument of the form:

$$p \rightarrow q$$

But p.

Therefore q.

Proof by Contradiction:

$$(17) (p \rightarrow q) \wedge \neg q \rightarrow \neg p$$

Example 7

“If you’d left the keys on the table they’d still be there. But the keys aren’t there so you couldn’t have left them there.”

This analyses formally to the following:

p = “You left your keys on the table.”

q = “The keys are still there.”

If p then q . But not q . Therefore not p .

(Of course there are hidden assumptions such as “nobody else has come in” and “keys cannot spontaneously evaporate”).

Transitive Property of Implication:

$$(18) (p \rightarrow q) \wedge (q \rightarrow r) \rightarrow (p \rightarrow r)$$

Note how the parentheses around $p \rightarrow r$ are necessary. A construction such as $p \rightarrow q \rightarrow r$ is ambiguous since implication is not associative.

§1.8. Quantifiers

A **predicate** is a statement that involves variables. Predicates become propositions when particular objects (e.g. numbers) are substituted for the variables. The resulting propositions have truth values that depend on those elements.

An **n -ary predicate** is one which applies to a combination of n elements. Special terms are **unary** if $n = 1$, **binary** if $n = 2$ and **ternary** if $n = 3$. A **unary** predicate is what we usually think of as a **property**, such as ‘ x is even’ or ‘ x is female’. We could write these symbolically as Ex for ‘ x is even’ and Fx for ‘ x is female’. There must be some underlying set over which the variables range. In the case of E it might be the set of integers and in the case of F it might be the set of all students in a particular class.

A **binary** predicate is what we usually think of as a relation, such as ‘ $x < y$ ’ or ‘ x has passed y ’, where x is a student and y is a course. We could write these symbolically as xLy for ‘ $x < y$ ’ and xPy for ‘ x has passed the course y ’. The first of these example might have x and y ranging over the set of integers and the second example might have x ranging over the set of students in a particular class and y ranging over the courses at a particular university.

Example 8

‘ x is even’ is a unary predicate that is TRUE for $x = 2$ and FALSE for $x = 3$.

‘ x loves y ’ is a binary predicate involving two elements x and y from a set of people.

If $x = \text{Romeo}$ and $y = \text{Juliet}$ then this becomes a TRUE statement.

' x is the distance between P and Q ' is a ternary predicate involving a combination of a number and two points.

Notation: We commonly denote predicates by upper case letters and the variables by lower case letters. Often the variable names follow the predicate name such as Pn , Qxy and $Rabc$. But in the case of a binary predicate it's more usual to put one element on each side of the predicate name as in xRy . This is what we usually do in mathematics with predicates such as '=' and '<'.



If P is a unary predicate then $\forall x Px$ means 'for all x , Px (is true)' and $\exists x Px$ denotes 'for some x , Px ', or 'there exists x such that Px '. 'For all' is called the **universal quantifier** and 'for some' is called the

existential quantifier.

NOTES:

(1) 'Some' means 'at least one'. Even if Px is true for just one x we're entitled to claim that $\exists x Px$.

(2) There's some assumed non-empty universe over which we are quantifying.

(3) The variable ‘bound’ by a quantifier is a ‘dummy’ variable and can be replaced by any other provided this is done consistently throughout the scope of the quantifier.

This means that $\forall x(Px \rightarrow Qx)$ is equivalent to

$$\forall t(Pt \rightarrow Qt) \text{ or}$$

$$\forall w(Pw \rightarrow Qw).$$

This is analogous to the dummy variable of integration in a definite integral and to local variables in many computing languages.

(4) When working with quantifiers, statements of the form ‘ Px implies Qx ’ usually have an implied universal quantifier and should be translated as $\forall x(Px \rightarrow Qx)$. It is for this reason that implication is defined to be TRUE when the ‘antecedent’ is FALSE.

The statement $\forall x(x \text{ is even} \rightarrow x^2 \text{ is even})$ has to be true for *all* integers, including odd ones. The definition of implication makes it ‘vacuously true’ in the case of odd integers.

(5) In normal language ‘implies’ often has causal overtones, but in logic this is not the case. The statement “I am male implies that I am intelligent” might seem to be saying that men are more intelligent than women. But if you are female, whether or not you are intelligent, the statement of the implication is TRUE.

Suppose that in a survey of 100 men, all who eat red meat happen to have good eyesight. A medical

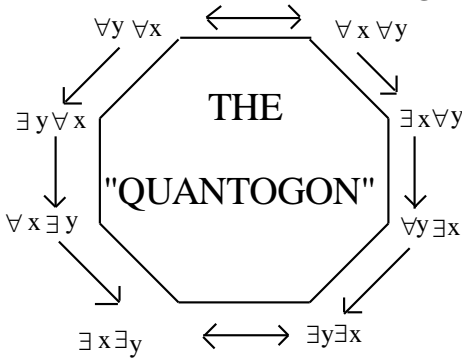
researcher might begin to investigate whether this is a coincidence, whether red meat is good for eyesight, whether good eyesight predisposes someone to eat red meat, or whether there is some other factor that causes both good eyesight and a liking for red meat.

For a logician the work is done. The assertion:

$$\forall m (m \text{ eats red meat} \rightarrow m \text{ has good eyesight})$$

is TRUE, provided we're limiting the universe of quantification to the above sample and are not extrapolating to the wider population.

§1.9. The Order of Quantifiers



With a predicate involving two variables there are eight ways they can be quantified: each of x, y can be quantified by \forall or \exists and they can be quantified in either order. The logical relationships

between them are displayed by the above diagram:

Note carefully the difference between $\exists x\forall y$ and $\forall y\exists x$. The first is the stronger of the two. It implies the second but not conversely. For example $\exists x\forall y[x > y]$ claims that there's an integer larger than every integer (FALSE) while $\forall y\exists x[x > y]$ makes the weaker (TRUE) claim that for every integer there's a larger one.

If xLy means ‘ x loves y ’ (within an appropriate universe) $\forall y \exists x [xLy]$ makes the claim that “nobody is unloved” while $\exists x \forall y [xLy]$ comes close to making the religious claim “There is a being who loves everybody”.

The definition of limits in calculus makes subtle use of quantifiers. The statement that ‘the limit of $f(x)$ as $x \rightarrow a$ is b ’ is defined as:

$$\forall \epsilon \exists \delta \forall x [(\epsilon > 0) \wedge (0 < |x - a| < \delta) \rightarrow (|f(x) - b| < \epsilon)].$$

If we interchanged the first two quantifiers we’d get the stronger statement:

$$\exists \delta \forall \epsilon \forall x [(\epsilon > 0) \wedge (0 < |x - a| < \delta) \rightarrow (|f(x) - b| < \epsilon)].$$

which, in effect, claims that $f(x) = b$ for all x in some region about $x = a$. This is not the same thing.

§1.10. Quantifiers in Mathematics

Suppose that the universe of quantification is

$$\mathbb{N} = \{0, 1, 2, \dots\},$$

the set of natural numbers and that the primitive predicate is $x = y$, supplemented by the functions of addition and multiplication. Other predicates can be built up from these as follows:

THIS	MEANS THAT
$x \neq y$	$\neg(x = y)$
x is even	$\exists y (x = 2y)$
$x \leq y$	$\exists z (y = x + z)$

THIS	MEANS THAT
$x < y$	$(x \leq y) \wedge (x \neq y)$ i.e. $\exists z (y = x + z) \wedge \neg (x = y)$
x divides y	$\exists z (y = xz)$
x is prime	$(x \neq 1) \wedge \forall d [d \text{ divides } x \rightarrow$ $(d = 1) \vee (d = x)]$ i.e. $\neg(x = 1) \wedge \forall y \exists d [x = yd$ $\rightarrow (d = 1) \vee (d = x)]$
x is a prime power	$\exists p [p \text{ is prime} \wedge \forall q [(q \text{ is prime})$ $\wedge (q \text{ divides } x) \rightarrow (q = p)]]$

In the last example this is a formal translation of the statement: ‘there’s at most only one prime divisor of x . So, technically, it includes 1 as a prime power, as it should do.

You might be confused by the fact that 1 is not a prime, yet it is a prime power, but remember that $1 = 2^0$.

The last statements can be expanded until it is expressed entirely in terms of equality, addition and multiplication.

The famous *Goldbach Conjecture* (whose truth is strongly suspected but has never been proved) asserts that “every even number bigger than 2 is the sum of two primes”.

This can be written as: $\forall x [(x > 2) \wedge (x \text{ is even}) \rightarrow \exists y \exists z [(y \text{ is prime}) \wedge (z \text{ is prime}) \wedge (x = y + z)]$

§1.11. Negation Rules



To prove a theorem by contradiction requires the negation of the theorem to be assumed in order to reach a contradiction. If the statement has a complicated logical structure it may be necessary to rewrite this negation more simply. One can do this by using the following rules describing the way negation interacts with the other truth operators and the two quantifiers.

structure it may be necessary to rewrite this negation more simply. One can do this by using the following rules describing the way negation interacts with the other truth operators and the two quantifiers.

PROPOSITION NEGATION

$\neg p$	p
$p \wedge q$	$\neg p \vee \neg q$
$p \vee q$	$\neg p \wedge \neg q$
$p \rightarrow q$	$p \wedge \neg q$
$p \leftrightarrow q$	$p \leftrightarrow \neg q$
$\forall x Px$	$\exists x \neg Px$
$\exists x Px$	$\forall x \neg Px$

Example 9

The negation of $\exists x [Px \rightarrow (Qx \wedge \neg Rx)]$ is

$$\begin{aligned} \neg \exists x [Px \rightarrow (Qx \wedge \neg Rx)] &\leftrightarrow \forall x \neg [Px \rightarrow (Qx \wedge \neg Rx)] \\ &\leftrightarrow \forall x [Px \wedge \neg (Qx \wedge \neg Rx)] \\ &\leftrightarrow \forall x [Px \wedge (\neg Qx \vee \neg \neg Rx)] \\ &\leftrightarrow \forall x [Px \wedge (\neg Qx \vee Rx)]. \end{aligned}$$

EXERCISES FOR CHAPTER 1

Exercise 1: Use a Truth Table to prove that:

$\neg q \vee (\neg p \rightarrow (p \leftrightarrow q))$ is a tautology.

Exercise 2: Use a Truth Table to prove that:

$(p \wedge \neg q) \rightarrow \neg(p \rightarrow q)$ is a tautology.

Exercise 3: Use a Truth Table to prove that:

$(p \rightarrow (q \rightarrow r)) \leftrightarrow \neg(p \wedge q) \vee r$ is a tautology.

Exercise 4: Use a Truth Table to determine whether or not the statement $p \wedge q \rightarrow \neg q \rightarrow p$ is a tautology.

Exercise 5: Use a Truth Table to determine whether or not the statement $((p \rightarrow q) \rightarrow p) \rightarrow q$ is a tautology.

Exercise 6: Use a Truth Table to show that the following statement is a tautology:

$\neg [(p \leftrightarrow \neg q) \wedge (r \rightarrow p) \wedge (p \rightarrow q) \wedge r]$

Exercise 7: Use a Truth Table to prove that:

$\neg(p \rightarrow q \wedge \neg r) \leftrightarrow (p \wedge (q \rightarrow r))$ is a tautology.

Exercise 8: Are $(p \rightarrow q) \rightarrow \neg p$ and $\neg(p \wedge q)$ equivalent?

Exercise 9: Show that $(p \leftrightarrow \neg q) \wedge (r \rightarrow p)$ and $(p \rightarrow q) \wedge r$ are inconsistent.

Exercise 10: Is the statement:

$(r \rightarrow p \wedge q) \rightarrow ((r \vee s) \rightarrow (t \rightarrow p))$ a tautology?

[**HINT:** A Truth Table for this question would have 64 rows which is a bit much. Instead, try to make the statement FALSE. You do this by making the first bit TRUE and the second bit FALSE. Continue in the same way.]

Exercise 11: Prove that:

$((q \vee r) \rightarrow (p \wedge s)) \rightarrow (s \rightarrow (p \rightarrow (q \vee r)))$ is not a tautology.

Exercise 12: Using Truth Tables, or otherwise, determine which of the following are tautologies.

(a) $(p \wedge q) \rightarrow (p \rightarrow q)$

(b) $\neg(p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$

(c) $p \wedge \neg(q \wedge r) \leftrightarrow \neg(p \rightarrow q) \vee (p \wedge \neg r)$

(d) $((r \vee s) \rightarrow (p \wedge q)) \rightarrow (p \rightarrow (q \rightarrow (r \vee s)))$

Exercise 13: Use a Truth Table to prove that:

$\neg(p \rightarrow q) \vee (p \wedge q) \leftrightarrow p$ is a tautology.

Exercise 14: Use a Truth Table to prove that:

$(p \wedge q) \vee \neg(p \rightarrow q) \leftrightarrow p$ is a tautology.

Exercise 15: *If you're not staying on Great Keppel Island you're not having a perfect holiday.*

Express this in symbols. What does it say about the island?

Exercise 16: Write out in symbols the sentence: ‘exactly one of p , q and r is true’.

Exercise 17:

(a) Let Pxy denote the statement $xyx = x$. If the universe of quantification is the set of all real numbers, which of the following are TRUE?

- (i) $\forall x \exists y Pxy$;
- (ii) $\exists x \forall y Pxy$;
- (iii) $\exists y \forall x Pxy$;
- (iv) $\forall x \forall y [Pxy \rightarrow Pyx]$;
- (v) $\exists x Pxx$.

(b) Repeat where the universe of quantification is the set of positive real numbers.

(c) Repeat when the universe of quantification is the set of positive integers.

Exercise 18: Explain in words why the statement: $x > y$ (for real numbers) can be written as:

$$\forall z [-(y = x + zz)].$$

Exercise 19: Express the following statement about real numbers symbolically, using only logical symbols and equations (no words or inequalities):

for all positive real numbers there is a smaller one

[NOTE: The only symbols you are allowed to use are the logical symbols, variables, brackets, ‘+’ and ‘=’. You may *not* use any other symbols such as \mathbb{R}^+ or inequalities. The

Universe of Quantification is \mathbb{R} , which you don't have to mention explicitly.]

Exercise 20: Express the statement ‘ m and n have no common factor’ in symbolic form using only logical symbols and primitive arithmetic statements of the form $x = yz$. (The universe of quantification is the set of positive integers.)

Exercise 21: Which of the following are true? Give brief reasons.

- (i) $\forall x \exists y [y < x]$ where the universe of quantification is \mathbb{R}^+ the set of all positive real numbers;
- (ii) $\exists y \forall x [y < x]$ where the universe of quantification is \mathbb{R}^+ , the set of all positive real numbers.

Exercise 22: Which of the following are TRUE, where the universe of quantification is \mathbb{Z} the set of all integers? Give brief reasons.

- (i) $\forall x \exists y [x + y = 100]$
- (ii) $\exists y \forall x [x + y = 100]$
- (iii) $\forall x \exists y [x y + x^2 = 0]$
- (iv) $\exists y \forall x [x^2 y < x + 1]$

Exercise 23: Negate the following statement, expressing the negation in its simplest form:

$$\forall x [Px \rightarrow \exists y [xQy \vee -yQx]]$$

Exercise 24: Negate the following statement, expressing the negation in its simplest form:

$$\exists x [xAx \rightarrow \forall y[-xAy \vee yAx]]$$

Exercise 25: Negate the following:

$$\forall x[Px \rightarrow \exists y(Qxy \wedge \neg Py)].$$

Exercise 26: Express the negation of:

$$\exists y[Py \rightarrow \forall z[\neg yQz \wedge Pz]]$$

in such a way that the negation operator is only attached to primitive statements such as Px or xQy . Now write this in an equivalent, but simpler, form which doesn't use the negation operator at all.

SOLUTIONS FOR CHAPTER 1

Exercise 1: It is a tautology since it is TRUE in all cases.

\neg	q	\vee	$(\neg$	p	\rightarrow	$(p$	\leftrightarrow	$q)$
F	T	T	F	T	T	T	T	T
T	F	T	T	T	F	T	F	F
F	T	T	T	F	T	F	F	T
T	F	T	T	F	T	F	T	F

Exercise 2: It is a tautology because it is always TRUE.

p	\wedge	\neg	$q)$	\rightarrow	\neg	$(p$	\rightarrow	$q)$
T	F	F	T	T	F	T	T	T
T	T	T	F	T	T	T	F	F
F	F	F	T	T	F	F	T	T
F	F	T	F	T	F	F	T	F

Exercise 3: It's a tautology since it's TRUE in all cases.

$$((p \rightarrow (q \rightarrow r)) \leftrightarrow (\neg (p \wedge q) \vee r))$$

T	T	T	T	T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	T	T	T	F	F
T	T	F	T	T	T	T	T	F	F	T	T
T	T	F	T	F	T	T	T	F	F	T	F
F	T	T	T	T	T	T	F	F	T	T	T
F	T	T	F	F	T	T	F	F	T	T	F
F	T	F	T	T	T	T	F	F	F	T	T
F	T	F	T	F	T	T	F	F	F	T	F

Exercise 4: It is a tautology since it is TRUE in all cases.

$$(p \wedge q) \rightarrow (\neg q \rightarrow p)$$

T	T	T	T	F	T	T	T
T	F	F	T	T	F	T	T
F	F	T	T	F	T	T	F
F	F	F	T	T	F	F	F

Exercise 5: It is not a tautology since it is FALSE when $p = T$ and $q = F$.

$$((p \rightarrow q) \rightarrow p) \rightarrow q$$

T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	T	T	F	F	T	T
F	T	F	F	F	T	F

Exercise 6: It is a tautology since it is TRUE in all cases.

$$- [((p \leftrightarrow -q) \wedge (r \rightarrow p)) \wedge ((r \rightarrow q) \wedge p)]$$

T	T	F	F	T	F	T	T	T	F	T	T	T	T	T
T	T	F	F	T	F	F	T	T	F	T	T	T	F	F
T	T	T	T	F	T	T	T	T	F	T	F	F	F	T
T	T	T	T	F	T	F	T	T	F	T	F	F	F	F
T	F	T	F	T	F	T	F	F	F	F	T	T	T	T
T	F	T	F	T	T	F	T	F	F	F	T	T	F	F
T	F	F	T	F	F	T	F	F	F	F	T	F	T	T
T	F	F	T	F	F	F	T	F	F	F	T	F	F	F

Exercise 7: It is a tautology since it is TRUE in all cases.

$$- (p \rightarrow (q \wedge -r)) \leftrightarrow ((p \wedge (q \rightarrow r)))$$

T	T	F	T	F	F	T	T	T	T	T	T	T	T
F	T	T	T	T	T	F	T	T	F	T	F	F	F
T	T	F	F	F	F	T	T	T	T	F	T	T	T
T	T	F	F	F	T	F	T	T	T	F	T	F	F
F	F	T	T	F	F	T	T	F	F	T	T	T	T
F	F	T	T	T	T	F	T	F	F	T	F	F	F
F	F	T	F	F	F	T	T	F	F	F	T	T	T
F	F	T	F	F	T	F	T	F	F	F	T	F	F

Exercise 8: Statements S_1 and S_2 are equivalent if and only if $S_1 \leftrightarrow S_2$ is a tautology. It is not, so they are not equivalent.

$$((p \rightarrow q) \rightarrow \neg p) \leftrightarrow (\neg (p \wedge q))$$

T	T	T	F	F	T	T	F	T	F	T
T	F	F	T	F	T	F	F	T	F	F
F	T	T	T	T	F	T	T	F	T	T
F	T	F	T	T	F	T	T	F	F	F

Exercise 9: Statements S_1 and S_2 are inconsistent if and only if $\neg (S_1 \wedge S_2)$ is a tautology.

This was shown in Exercise 6. So they are inconsistent.

Exercise 10: Let us try to find a case which makes this FALSE without having to consider all 16 possibilities.

We put F under the main \rightarrow and work backwards.

$$(r \rightarrow (p \wedge q)) \rightarrow ((r \vee s) \rightarrow (t \rightarrow p))$$

	T				F		T		F	T	F	F
--	---	--	--	--	----------	--	---	--	---	---	---	---

So $t = T$ and $p = F$. Now working forwards, as far as we can, we get:

$$(r \rightarrow (p \wedge q)) \rightarrow ((r \vee s) \rightarrow (t \rightarrow p))$$

F	T	F	F		F		T		F	T	F	F
---	---	---	---	--	---	--	---	--	---	---	---	---

(Since p is FALSE, $p \wedge q$ must be FALSE. But $r \rightarrow (p \wedge q)$ is TRUE, so r is FALSE.)

Transferring this across and continuing we get:

$$(r \rightarrow (p \wedge q)) \rightarrow ((r \vee s) \rightarrow (t \rightarrow p))$$

F	T	F	F		F	F	T	T	F	T	F	F
---	---	---	---	--	---	---	---	---	---	---	---	---

So $p = F$, $r = F$, $s = T$ and $t = T$. We can't deduce the truth value of q . All this means is that there are two cases

which make the statement FALSE, one where $q = T$ and one where it is FALSE. The fact that there is at least one case which makes the statement FALSE shows that it is not a tautology.

Exercise 11: Let us try to home in on a case which makes this FALSE without having to consider all 16 possibilities. We put F under the main \rightarrow and work backwards.

$$((\quad \vee r \rightarrow (\quad \wedge s) \rightarrow (\quad \rightarrow (\quad \rightarrow (\quad \vee r)))$$

$$q \quad \quad \quad p \quad \quad \quad s \quad \quad p \quad \quad q \quad \quad)$$

			T				F	T	F	T	F	F	F	F
--	--	--	---	--	--	--	---	---	---	---	---	---	---	---

So p, s are TRUE and q, r are FALSE. Now working forwards we get:

$$((\quad \vee r \rightarrow (\quad \wedge s) \rightarrow (\quad \rightarrow (\quad \rightarrow (\quad \vee r)))$$

$$q \quad \quad \quad p \quad \quad \quad s \quad \quad p \quad \quad q \quad \quad)$$

F	F	F	T	T	T	T	F	T	F	T	F	F	F	F
---	---	---	---	---	---	---	---	---	---	---	---	---	---	---

There's a consistent way of filling out this row so it's not a tautology.

Exercise 12: (a) is a tautology.

$$(p \wedge q) \rightarrow (p \rightarrow q)$$

T	T	T	T	T	T	T
T	F	F	T	T	F	F
F	F	T	T	F	T	T
F	F	F	T	F	T	F

(b) is NOT a tautology.

$$\neg (p \rightarrow q) \leftrightarrow (\neg p \rightarrow \neg q)$$

F	T	T	T	F	F	T	T	F	T
T	T	F	F	T	F	T	T	T	F
F	F	T	T	T	T	F	F	F	T
F	F	T	F	F	T	F	T	T	F

(c) is a tautology.

$$(p \wedge \neg (q \wedge r)) \leftrightarrow (\neg p \rightarrow q \vee (p \wedge \neg r))$$

T	F	F	T	T	T	T	F	T	T	T	F	T	F	F	T
T	T	T	T	F	F	T	F	T	T	T	T	T	T	T	F
T	T	T	F	F	T	T	T	T	F	F	T	T	F	F	T
T	T	T	F	F	F	T	T	T	F	F	T	T	T	T	F
F	F	F	T	T	T	T	F	F	T	T	F	F	F	F	T
F	F	T	T	F	F	T	F	F	T	T	F	F	F	T	F
F	F	T	F	F	T	T	F	F	T	F	F	F	F	F	T
F	F	T	F	F	F	T	F	F	T	F	F	F	F	T	F

(d) Let's try to find a case where it is FALSE.

If it is FALSE then $(r \vee s) \rightarrow (p \wedge q)$ is TRUE and $p \rightarrow (q \rightarrow (r \vee s))$ is FALSE.

From the latter, $p = T$ and $q \rightarrow (r \vee s) = F$.

Thus $q = T$ and $r \vee s = F$.

Thus r, s are both FALSE.

So in the case $p = T, q = T, r = F, s = F$ the statement is FALSE. Hence it is not a tautology.

Exercise 13: is a tautology

$$[\neg (p \rightarrow q) \vee (p \wedge q)] \leftrightarrow p$$

F	T	T	T	T	T	T	T	T	T
T	T	F	F	T	T	F	F	T	T
F	F	T	T	F	F	F	T	T	F
F	F	T	F	F	F	F	F	T	F

Exercise 14: is a tautology

$$[(p \wedge q) \vee \neg (p \rightarrow q)] \leftrightarrow p$$

T	T	T	T	F	T	T	T	T	T
T	F	F	T	T	T	F	F	T	T
F	F	T	F	F	F	T	T	T	F
F	F	F	F	F	F	T	F	T	F

Exercise 15:

Let $g =$ ‘staying on Great Keppel Island’ and
 let $p =$ ‘having a perfect holiday’.

The statement says that $\neg g \rightarrow \neg p$ which is equivalent to $p \rightarrow g$. It says you won't have a perfect holiday anywhere else, but makes no claim about Great Keppel itself. Perhaps there's no such thing as a perfect holiday anywhere!

Exercise 16: $(p \vee q \vee r) \wedge \neg(p \wedge q) \wedge \neg(p \wedge r) \wedge \neg(q \wedge r)$

Exercise 17:

(a) (i) is TRUE. If $x \neq 0$, let $y = 1/x$.

If $x = 0$, let $y =$ anything. Then $xyx = x$.

(ii) is TRUE. Take $x = 0$.

(iii) is FALSE. Putting $x = 1$ gives $y = 1$. Putting $x = 2$ gives $y = 1/4$.

So there is no such y that works for all x .

(iv) is FALSE. Let $x = 0$ and $y = 1$. Then $xyx = x = 0$, but $xyx = 0 \neq y$.

(v) is TRUE. Let $x = 1$.

(b) (i) is TRUE. Let $y = 1/x$.

(ii) is FALSE, for taking $x = 1$ gives $y = 1$. But this would mean that $x^2 = x$ for all $x > 0$, which is not so.

(iii) is FALSE, as above.

(iv) is TRUE. If $xyx = x$ then $y = 1/x$ and $xyx = y$.

(v) is TRUE as above.

(c) (i) is FALSE. Take $x = 2$. The corresponding y would have to satisfy $4y = 2$, which has no solution in the positive integers.

(ii) is FALSE, as above.

(iii) is FALSE since (i) is FALSE.

(iv) is TRUE. If $xyx = x$ then $xy = 1$ and so $x = y = 1$ (only solution for positive integers). Thus $xyx = y$.

(v) is TRUE as above.

Exercise 18: $(x > y) \leftrightarrow (y - x < 0)$. Negative real numbers are precisely those that have no square roots, so $x < 0$ can be written as $\neg \exists z[x = zz]$. So we can write the above as:

$$x > y \leftrightarrow \neg \exists z [y - x = zz] \leftrightarrow \forall z [-(y - x = zz)] \\ \leftrightarrow \forall z [-(y = x + zz)].$$

Exercise 19: The statement can be expressed as:

$$\forall x [x > 0 \rightarrow \exists y[(y > 0) \wedge (y < x)]].$$

We can write it in more primitive terms as:

$$\forall x [\forall z [-(x + zz = 0)] \rightarrow \\ \exists y [\forall z [-(y + zz = 0)] \wedge \forall z [-(y = x + zz)]]].$$

Exercise 20:

The statement can be expressed as:

$$\forall d [((d \text{ divides } m) \wedge (d \text{ divides } n)) \rightarrow (d = 1)].$$

This can be expressed in more primitive terms as:

$$\forall d [(\exists q[m = dq] \wedge \exists q[n = dq]) \rightarrow (d = 1)]$$

NOTE: Since q is only a ‘local variable’ in each place we are able to use the same symbol each time, even though the q that works for m will generally be different to that which works for n .

Exercise 21:

(i) is TRUE. For example take $y = x/2$.

(ii) is FALSE. If there was such a y , then taking $x = y$ we’d get $y < y$ which is impossible.

Exercise 22: (i) is TRUE. Take $y = 100 - x$.

(ii) is FALSE. If it was TRUE then all integers would be equal ($= 100 - y$).

(iii) is TRUE. Take $y = -x$.

(iv) is TRUE. Take $y = -1$.

Then $x^2y - x - 1 = -(x^2 + x + 1)$ which is always negative.

Exercise 23: The negation is:

$$\begin{aligned} & \neg \forall x [Px \rightarrow \exists y [xQy \vee \neg yQx]] \\ & \quad \leftrightarrow \exists x [\neg [Px \rightarrow \exists y [xQy \vee \neg yQx]]] \\ & \quad \leftrightarrow \exists x [Px \wedge \neg \exists y [xQy \vee \neg yQx]] \\ & \quad \leftrightarrow \exists x [Px \wedge \forall y [\neg [xQy \vee \neg yQx]]] \\ & \quad \leftrightarrow \exists x [Px \wedge \forall y [\neg xQy \wedge yQx]] \end{aligned}$$

Exercise 24: The negation is:

$$\begin{aligned} & \neg \exists x [xAx \rightarrow \forall y [\neg xAy \vee yAx]] \\ & \quad \leftrightarrow \forall x \neg [xAx \rightarrow \forall y [\neg xAy \vee yAx]] \\ & \quad \leftrightarrow \forall x [xAx \wedge \neg \forall y [\neg xAy \vee yAx]] \\ & \quad \leftrightarrow \forall x [xAx \wedge \exists y \neg [\neg xAy \vee yAx]] \\ & \quad \leftrightarrow \forall x [xAx \wedge \exists y [xAy \wedge \neg yAx]] \end{aligned}$$

Exercise 25: The negation is:

$$\begin{aligned} & \neg \forall x [Px \rightarrow \exists y (Qxy \wedge \neg Py)] \\ & \quad \leftrightarrow \exists x \neg [Px \rightarrow \exists y (Qxy \wedge \neg Py)] \\ & \quad \leftrightarrow \exists x [Px \wedge \neg \exists y (Qxy \wedge \neg Py)] \\ & \quad \leftrightarrow \exists x [Px \wedge \forall y \neg (Qxy \wedge \neg Py)] \\ & \quad \leftrightarrow \exists x [Px \wedge \forall y (\neg Qxy \vee Py)]. \end{aligned}$$

Exercise 26: $\neg \exists y [Py \rightarrow \forall z [\neg yQz \wedge Pz]]$

$$\begin{aligned} & \leftrightarrow \forall y \neg [Py \rightarrow \forall z [\neg yQz \wedge Pz]] \\ & \quad \leftrightarrow \forall y [Py \wedge \neg \forall z [\neg yQz \wedge Pz]] \end{aligned}$$

$$\begin{aligned}
&\leftrightarrow \forall y [Py \wedge \rightarrow \exists z \neg[\neg yQz \wedge Pz]] \\
&\leftrightarrow \forall y [Py \wedge \rightarrow \exists z [yQz \vee \neg Pz]] \\
&\leftrightarrow \forall y [Py \wedge \rightarrow \exists z [yQz]].
\end{aligned}$$

This last step follows because, if Py is true for all y then in particular so is Pz and so the $\neg Pz$ option is redundant.