

4. SETS, FUNCTIONS & RELATIONS

*If I see the moon, then the moon sees me
'Cos seeing's symmetric as you can see.
If I tell Aunt Maude and Maude tells the nation
Then I've told the nation
'cos the gossiping relation
Is transitive. And even if you're on the shelf
You're loved by someone,
that's yourself,
'cos love's reflexive. And
if all three,
Reflex- and Trans- and
Symmetry,
I'm sure you see, it makes
good sense
To call it a relation of equivalence.*



*Ah, when I was young it came to pass
That I learnt all this in equivalence class.*

§4.1. Sets in Maths and Computing

As we studied elementary mathematics we came to believe that the fundamental object in mathematics is the number. Then, as we began calculus, we learnt of a new type of mathematical object — the function. At first we thought of a function as a formula, or an algebraic

expression, such as the function x^2 . But then they tried to tell us that functions are more than just formulae. They are rules which associate with every element in one set a unique element of another. Every function has to go from one set to another.

The deeper we get into mathematics the more we become involved with sets. We learn of vector spaces and groups — these are sets on which there is some algebraic structure. In geometry we no longer deal with shapes, but rather, sets of points. And in calculus we're increasingly concerned with domains of functions and regions of convergence and sets of solutions to differential equations. The set is paramount in mathematics.

In computing science sets are one of many important data structures. In databases they are the fundamental data type.

§4.2. Defining a Set

A **set** is a collection of objects called **elements**. We write $x \in S$ if x is an element of the set S and $x \notin S$ if it is not.

Generally elements are denoted by lower case letters and sets by capitals. However since there are sets whose elements are sets, and sets of sets of sets etc., it's not always possible to maintain this distinction.

There are three main ways of describing or defining a set:

(1) Use a standard name

\mathbb{N} = the set of natural numbers 0, 1, 2,

NOTE: Some books exclude zero from this set and consider the natural numbers to be synonymous with the positive integers.

Example 1:

\mathbb{Z} = the set of integers $0, \pm 1, \pm 2, \dots$ (from the German ‘Zahlen’, meaning ‘numbers’);

\mathbb{Q} = the set of rational numbers (\mathbb{Q} stands for ‘quotient’);

\mathbb{R} = the set of real numbers;

\mathbb{C} = the set of complex numbers.

(2) List the elements

$\{x_1, x_2, \dots, x_n\}$ denotes the set with the elements:

$$x_1, x_2, \dots, x_n.$$

Strictly speaking this notation can only be used for finite sets but if there’s an obvious pattern we can indicate certain infinite sets this way.

Example 2:

$\{4, 5, 13\}$ is the set whose elements are the integers 4, 5 and 13;

$\{0, 1, 4, 9, \dots\}$ indicates the set of square integers.

(3) Describe a property that characterises the elements.

$\{x \in S \mid Px\}$ denotes the set of all elements x , in S , for which the statement Px is true. If the set S is understood we often omit it and write $\{x \mid Px\}$.

Example 3:

$\{x \in \mathbb{Z} \mid x > 0\}$ is the set of positive integers, commonly denoted by \mathbb{Z}^+ ;

$\{x \in \mathbb{Z} \mid 3 < x < 7\} = \{4, 5, 6\}$;

$\{x \in \mathbb{R} \mid x^2 < 1\}$ denotes the open interval $(-1, 1)$ i.e. all real numbers x such that $-1 < x < 1$;

$\{n \mid \exists q[n = 7q]\}$ denotes the set of all multiples of 7.

NOTES:

(1) The symbol x in the notation $\{x \mid Px\}$ is a ‘dummy’ variable. It can be replaced throughout by any other symbol not otherwise used. Thus $\{x \mid Px\} = \{r \mid Pr\}$.

(2) $y \in \{x \mid Px\} \leftrightarrow Py$, that is, y belongs to a set if and only if it has its defining property.

(3) When we describe a finite set by listing the elements the order doesn’t matter. Also any repetitions are ignored. For example:

$$\{3, 1, 2, 4\} = \{1, 2, 3, 4\} = \{1, 2, 2, 3, 4, 4, 4\}.$$

(4) It was once naively thought that for every property there must be a set, but this can lead to certain paradoxes. The most famous is the Russell Paradox named after the philosopher and mathematician Bertrand Russell. If $S = \{x \mid x \notin x\}$ then $S \in S$ if and only if $S \notin S$. [If $S \in S$ it satisfies the defining property and so $S \notin S$. If $S \notin S$ it must be because it doesn’t satisfy the defining property and so $S \notin S$ is FALSE and so $S \in S$!] We clearly can’t

allow a mathematical system to exist in which such paradoxes are possible. Some time ago, some mathematicians interested in the foundations of mathematics established axioms for set theory in which there are precise restrictions on which properties are allowed to give rise to a set. Fortunately such deep problems are far removed from the coal-face of useful mathematics and we can safely ignore them.

Two sets are defined to be **equal** if every element of one set is an element of the other. In symbols:

$$S = T \leftrightarrow \forall x[x \in S \leftrightarrow x \in T]$$

Example 4: $\{x \in \mathbb{Z} \mid x^2 < 2\} = \{-1, 0, 1\}$
 $= \{x \in \mathbb{R} \mid x^3 = x\}$

A set S is a **subset** of a set T if every element of S is an element of T . We write $S \subseteq T$ to denote the fact that S is a subset of T .

In symbols this is very similar to the definition of equality of sets:

$$S \subseteq T \leftrightarrow \forall x [x \in S \rightarrow x \in T]$$

In particular every set is a subset of itself. Subsets S which are not the whole set T are called **proper** subsets. We denote this by $S \subset T$.

A very important set is the **empty** set. This is the set with no elements and can be described by an empty list $\{ \}$ or

an impossible property, such as $\{x \mid x \neq x\}$. Note that there is only one empty set. For example, the empty set of integers whose square is 2 is the same as the empty set of triangles with four sides. The statement “mermaids don’t exist” can be expressed by saying that “the set of mermaids is the empty set”. The empty set has its own special symbol, \emptyset . Like the number 0, it’s an extremely useful concept. If we want to prove that a given property P can never hold we can consider the set $S = \{x \mid Px\}$ and then prove that $S = \emptyset$.

Another important set is the ‘universe’ a set consisting of all the elements with which we are concerned in a given context. For example if we’re considering sets of integers we’d take our universe to be the set of *all* integers.

§4.3. Basic Set Functions

We now list a large number of definitions of functions which assign to each set S , or pair of sets S and T , a certain set.

Intersection:

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

Union:

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

Difference:

$$S - T = \{x \mid x \in S \text{ and } x \notin T\}$$

Complement:

$$-S = \{x \mid x \notin S\} = U - S$$

where U is the universe in which we're working.

Example 5

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$.

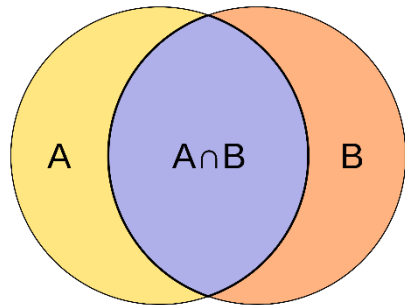
Then $S \cap T = \{1, 3\}$; $S \cup T = \{1, 2, 3, 5, 7\}$;

$S - T = \{2\}$; $T - S = \{5, 7\}$

If our universe is the set of all integers and S is the set of even numbers, $-S$ is the set of all odd numbers.

§4.4. Venn Diagrams

Some of these concepts can be illustrated by diagrams where sets are represented by regions drawn in the plane and elements by points inside them.



Traditionally Venn diagrams represent sets by overlapping ovals. The intersection, for example, is represented by the overlapped region.

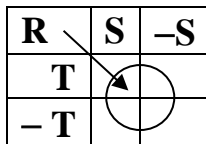
However, the simplest system is to divide a rectangle into two rows and two columns. The left side represents one set and the right side, its complement. Then the top half can represent a second set and the bottom, its complement.

	S	-S
T		
-T		

Example 6:

	S	-S		S	-S
T					
-T					
	$S \cap T$			$S \cup T$	

If there's a third set we can simply draw a circle in the middle, cutting across all four regions. The region inside the circle represents the third set and the outside represents its complement.



§4.5. Extended Set Functions

Cartesian Products:

$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}$
 = the set of all ordered pairs whose components
 come from S and T respectively

$$S \times T \times R = \{(x, y, z) \mid x \in S, y \in T, z \in R\} \text{ etc.}$$

Cartesian Power:

$$S^n = S \times S \times \dots \times S \text{ (} n \text{ factors)}$$

Power Set:

$$\wp(S) = \{A \mid A \subseteq S\} = \text{the set of all subsets of } S$$

Example 7:

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$. Then

$$S \times T = \{(1, 1), (1, 3), (1, 5), (1, 7), (2, 1), (2, 3), (2, 5), \\ (2, 7), (3, 1), (3, 3), (3, 5), (3, 7)\}$$

$$S \cap T = \{1, 3\}$$

$$(S \cap T)^3 = \{(1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3), \\ (3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3)\}$$

$$\wp(S) = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \\ \{1, 2, 3\}\}$$

NOTES:

(1) While the order of the elements of a set does not matter there's often a particular order that's more natural than the others. The advantage of using a systematic order is that it ensures that every combination has been accounted for. Look carefully at the order in which we've listed the elements in the above example.

(2) Venn Diagrams can't be used to depict these extended set functions.

§4.6. Relations

“Jack is my mother's uncle.” That is what we usually think of when we talk about relations — people with whom we are related. But it's the relationship itself that mathematicians would call a *relation*.

A **relation** is a statement involving two variables, or two gaps, where the names of people or things can be inserted.

So ‘is the mother of’ is one such relation. In mathematics there are numerous relations such as ‘is less than’, ‘is parallel to’, ‘is a power of’.



We can denote the fact that x has the relation R with y by xRy . This reflects the word order in natural language, and indeed where we've invented symbols for particular relations in mathematics we usually use this format, such as ‘ $x < y$ ’.

But a relation, as well as being a connection between two sets, can also be considered as a set in itself.

Suppose R is a relation such that xRy makes sense for $x \in S$ and $y \in T$. We can represent R by the set of ordered pairs (x, y) for which the relation holds, that is:

$$\{(x, y) \mid xRy\}.$$

Now $S \times T$ denotes the set of all ordered pairs (x, y) with $x \in S, y \in T$. So the set of ordered pairs for which xRy (is true) is just a subset of $S \times T$. This gives us an alternative definition of a relation.

A relation from a set S to a set T is a subset of $S \times T$.

A relation on a set S is a relation from S to S.

Example 8: If $S = \{1, 2, 3, 4\}$ then the relation normally written ' $x < y$ ' would be:

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

and the relation ' $x = y^n$ for some integer $n \geq 1$ ' becomes:

$$\{(1, 1), (2, 2), (3, 3), (4, 2), (4, 4)\}.$$

When a computer makes reference to a relation such as ' $<$ ' it certainly doesn't consult an array of ordered pairs. But when dealing with the relation 'student x is enrolled in course y ' we would store this information in a database as an array of ordered pairs.

Example 9:

STUDENT	COURSE
AARON Adam	MATHS 1
AARON Adam	PHYSICS 1
ANTON Tom	MATHS 1

In mathematical notation we'd write this relation as
 $\{(AARON\ Adam, MATHS\ 1),$
 $(AARON\ Adam, PHYSICS\ 1),$
 $(ANTON\ Tom, MATHS\ 1)\}$

§4.7. Equivalence Relations

Like all mathematical objects, relations can be classified according to whether they satisfy certain properties. The three most important properties for a relation R on a set S are the **reflexive**, **symmetric** and **transitive** properties.

- R is **reflexive** if xRx for all x .
- R is **symmetric** if $xRy \rightarrow yRx$ for all x, y .
- R is **transitive** if xRy and $yRz \rightarrow xRz$ for all x, y, z .

Example 14: $x < y$ on the set of integers.

Not Reflexive: For example $1 < 1$ is FALSE.

Not Symmetric: For example $1 < 2$ is TRUE but $2 < 1$ is FALSE.

Transitive: If $x < y$ and $y < z$ it follows that $x < z$.

Example 15: $xRy \leftrightarrow x = y^n$ for some $n \in \mathbb{N}$ is a relation on the set \mathbb{N} .

Reflexive: since $x = x^1$ for all x .

Not symmetric: Now $4R2$ (as $4 = 2^2$).

But it's not true that $2R4$, since $2 \neq 4^n$ for any positive integer n).

Transitive: Suppose xRy and yRz .

Then $x = y^n$ and $y = z^m$ for some $m, n \in \mathbb{N}$.

(We mustn't fall into the trap of assuming that the power is the same in each case.)

Hence $x = (z^m)^n = z^{mn}$. Since $mn \in \mathbb{N}$ it follows that xRz .

Example 16: $xRy \leftrightarrow |x - y| < 3$

Reflexive: Since for all x , $|x - x| = 0$ which is less than 3.

Symmetric: Since $|y - x| = |x - y|$.

Not Transitive: For example $1R3$ and $3R5$ but it's not true that $1R5$.

Example 17: $xRy \leftrightarrow x - y$ is even, defined on the set \mathbb{N} .

Reflexive : For all x , $x - x = 0$ which is even.

Symmetric: Suppose xRy .

This means that $x - y$ is even,

say $x - y = 2h$ for some $h \in \mathbb{Z}$.

Then $y - x = -2h = 2(-h)$ which is even, so yRx .

Transitive: Suppose xRy and yRz .

This means that $x - y = 2h$ for some $h \in \mathbb{Z}$ and

$y - z = 2k$ for some $k \in \mathbb{Z}$.

Now $x - z = (x - y) + (y - z) = 2h + 2k = 2(h + k)$ which is even.

Hence xRz .

An **equivalence relation** is a relation on a set that is reflexive, symmetric and transitive. Example 17 is an example of an equivalence relation.

§4.8. Equivalence Classes

If R is an equivalence relation and $x \in S$, we define

$$[x]_R = \{y \in S \mid xRy\},$$

that is, the set of all elements of S that are equivalent to x under the relation R .

We call $[x]_R$ the **equivalence class** containing x . Often we omit the subscript R .

NOTE: The fact that $[x]_R$ does indeed contain x follows from the reflexive property.

To find the equivalence classes for a given equivalence relation use the following simple algorithm. Choose an element x which hasn't yet been included in an equivalence class (any element will do to begin with). List all the elements that are related to x . These form an equivalence class. Now choose another element not yet listed. Continue until all elements have been included.

Equivalence classes are somewhat like families. Those in the same class are related – those in different classes are not. Human families don't provide a perfect analogy with equivalence classes since being related is not an equivalence relation. (The relation of being related is not transitive since my cousin has cousins who I wouldn't consider as being related to me.)

Example 18: Consider the set of railway stations of the world with xRy meaning that you can travel by rail from station x to station y (possibly having to change trains one or more times). The equivalence classes are the connected railway networks. One would be the government railways of mainland Australia. Another would have once been the Tasmanian Railways (they no longer exist). Some privately owned coal railways would form little

equivalence classes. Once the British Rail network in England, Scotland and Wales formed an entire equivalence class on its own, but with the opening of the Channel Tunnel it now forms part of a larger network that extends right across Europe.

Example 19: Let $S = \{1,2,3\}$ and let $R = \{(1,1), (1,3), (2,2), (3,1), (3,3)\}$.

One can show that R is an equivalence relation on S . The equivalence classes are:

$$[1] = \{1,3\} = [3] \text{ and } [2] = \{2\}.$$

Theorem 1: $x \in [y] \leftrightarrow [x] = [y]$

Proof: Suppose that $[x] = [y]$. Since $x \in [x]$, $x \in [y]$.

Conversely, suppose that $x \in [y]$.

Then yRx (and by the symmetric property, xRy).

We must show that $[x] = [y]$.

We first show that $[x] \subseteq [y]$.

Suppose that $z \in [x]$. Then xRz .

Since yRx , it follows from the transitive property that yRz and hence zRy by the symmetric property.

Thus $z \in [y]$, and so $[x] \subseteq [y]$.

Similarly, $[y] \subseteq [x]$, and so $[x] = [y]$. 🙌😊

Theorem 2: For all $x, y \in S$, either:

$$[x] \cap [y] = \emptyset \text{ or } [x] = [y].$$

Proof: Suppose that $[x], [y]$ are not disjoint.

Let $z \in [x] \cap [y]$.

Then $z \in [x]$ and $z \in [y]$.

By the previous theorem, $[x] = [z]$ and $[z] = [y]$ and so $[x] = [y]$. 🙌😊

We've proved that any two distinct equivalence classes are disjoint. So:

This means that S can be chopped up into non-overlapping equivalence classes so that elements are equivalent if and only if they belong to the same class.

**IF R IS AN EQUIVALENCE
RELATION ON A SET S THEN S IS
THE DISJOINT UNION OF ITS
EQUIVALENCE CLASSES**

§4.9. Functions

A **function** f is a pair of sets (S, T) together with a rule that associates, with each element $x \in S$, a unique element of T , written $f(x)$. This element is called the **image** of x under f .

We indicate that f is a function from S to T by writing $f: S \rightarrow T$. The set S is called the **domain** of f and T is called the **codomain**. Other words that are used instead of 'function' are 'map', 'operator' and 'transformation'.

Example 20: $f: P \rightarrow P$ where P is the set of people who are now living or who have died, with $f(x)$ defined to be

the father of x . But note that ‘the son of x ’ doesn’t define a function on P because, although everyone has a unique (genetic) father, not everyone has a unique son and many people in P have no sons at all.

The simplest way to describe a function from one finite set to another is by means of a table of values.

Example 21: $f: \{1, 2, 3, 4\} \rightarrow \{1, 2, 3, 4\}$ where f is defined by means of the following table of values:

x	$f(x)$
1	2
2	1
3	3
4	2

You don’t have to discover any pattern in this table, or find a formula, before you can call f a function. The rule here is simply to look up the table. The second column could have been filled up in any way, as long as no symbols other than 1, 2, 3, 4 is used.

Example 22:

There are 8 functions from $\{1, 2, 3\}$ to $\{A, B\}$ namely

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
1	A	A	A	A	B	B	B	B
2	A	A	B	B	A	A	B	B
3	A	B	A	B	A	B	A	B

Notice the systematic way in which these functions have been listed.

§4.10. Strings and Languages

If A is any set of symbols (it is usually finite and is called an **alphabet**) then a **string** on A is a finite sequence of symbols juxtaposed without any separators:

$$a_1 a_2 \dots, a_n.$$

Example 1: The following are some strings on the alphabet $\{1, 2, +, =\}$:

$$\alpha = '1+1=2',$$

$$\beta = '21+2122',$$

$$\gamma = '22+12=11',$$

$$\delta = '=+1+=+'$$

(The quote marks aren't part of the strings.)

String α represents a TRUE statement in ordinary arithmetic. String β is a valid arithmetic expression, but is not a statement. String γ is an arithmetic statement but is FALSE. String δ is pure nonsense. But *all* four are valid strings on the alphabet.



The **length** of the string $\alpha = a_1 a_2 \dots a_n$ is defined to be n and is denoted by $|\alpha|$.

Example 2: For the strings in example 1, $|\alpha| = 5$, $|\beta| = 7$, $|\gamma| = 8$, $|\delta| = 6$.

The **null** string is the unique string of length 0 and is denoted by the symbol λ .

It might seem a bit ridiculous to invent a word, and a symbol, for a string of no symbols. But people once thought that about the number zero. ‘Nothing’, whether it’s the number zero, the empty set, or the null string, is a very useful concept. Life would be so much more complicated without it.

The **product** of two strings

$$\alpha = a_1a_2 \dots a_m,$$

$$\beta = b_1b_2 \dots b_n,$$

is defined to be the string

$$\alpha\beta = a_1a_2 \dots a_mb_1b_2 \dots b_n.$$

This operation is called **concatenation**.

Example 3: Suppose α , β , γ and δ are the strings defined in example 1, and λ is the null string. Then $\alpha\delta = ‘1+1=2=+1+=+’$ while $\delta\alpha = ‘+=+1+=+1+1=2’$. Neither makes much sense but both are nevertheless valid strings.

Observe that $\alpha\delta \neq \delta\alpha$. In general, strings don’t commute. But note that $\alpha\lambda = \lambda\alpha$ since both give just ‘1+1=2’. In fact concatenating the null string on either end of *any* string leaves that string unchanged.

Properties: For all strings on a given alphabet:

$$(\alpha\beta)\gamma = \alpha(\beta\gamma)$$

$$\alpha\lambda = \lambda\alpha$$

$$|\alpha\beta| = |\alpha| + |\beta|$$

$$|\lambda| = 0$$

The set of all strings on a given alphabet is **closed** under the operation of concatenation, the **associative** law holds and there is an **identity**, namely the null string. However strings don't have **inverses** (apart from the null string which is its own inverse) because concatenation cannot make a string shorter.

An abstract structure in which there is a binary operation that satisfies both the closure law and the associative law, and which has an identity element, is known as a **semigroup**. A **group** is a semigroup in which every element has an inverse. The structure formed by the set of all strings on a given alphabet by the concatenation operation, is therefore a semigroup, but not a group because of the lack of inverses.

Normally we think of languages as special systems where certain strings are given meaning. There are the natural languages with which we communicate with one other in our daily lives such as English, French or Swahili. Then there are the specialised programming languages with which we communicate with computers such as BASIC, JAVA and C++.

But there are many other specialised systems that we do not usually think of as languages. Morse Code is a language that is built on top of a natural language. Chess enthusiasts and knitters have developed specialised languages in which they describe their activities.

There are two aspects to any language — *syntax* and *semantics*. Syntax is concerned with determining whether a given string has a valid form while semantics is concerned with the meanings that are given to acceptable strings. Semantics can only be discussed in the context of a particular language but syntax is more formal and can be discussed abstractly.

The sentence “The silver prejudice accelerated within the complicated sausage” is a syntactically correct English sentence. The verbs, nouns and adjectives are all in correct positions relative to one another. But semantically it is nonsense. It lacks meaning. On the other hand, the syntax of the sentence “Is near post-office where?” is incorrectly structured for English but, if a foreign tourist stopped you in the street with this question, you might be able to guess that he wanted to be directed to the nearest post-office. So correct syntax is not always necessary for communication. But it certainly makes communication much easier!

Decimal notation for real numbers is a mini-language that uses the ten digits, the minus sign and the decimal point.

The semantics of this language involves concepts such as powers of 10 and limits of sequences, but the syntax can be expressed by a few simple rules:

- (1) If a minus sign is included it must be at the front.
- (2) At most one decimal point may be included and if included it must have a digit on either side.
- (3) The first digit can't be 0 unless it's the only symbol or a decimal point follows.

The above examples are languages that are used in practice, but when one studies languages mathematically the word **language** refers to any (finite or infinite) set of strings. One of the fundamental problems is to describe an infinite language by giving a finite description. This is discussed in some depth in *Languages and Machines* in another set of notes.

EXERCISES FOR CHAPTER 4

Exercise 1:

Let $A = \{1, 3, 5, 9, 11\}$,

$B = \{n \in \mathbb{N} \mid n < 10\}$,

$C = \{n \in \mathbb{N} \mid n \text{ is prime}\}$,

$D = A \cap C$ and

$E = A - C$.

List the elements of:

- (a) B;
- (b) D;
- (c) E;

- (d) $B - A$;
- (e) $A \cup (B \cap C)$;
- (f) $D \times E$.

Exercise 2: Simplify each of the following (each answer is either S or \emptyset):

- (a) $S \cap S$;
- (b) $S - S$;
- (c) $S \cap \emptyset$;
- (d) $S \times \emptyset$;
- (e) $S \cup \emptyset$

Exercise 3: Which of the following statements are true for all sets R , S and T ?

- (a) $S \cap T = T \cap S$;
- (b) $S - T = T - S$;
- (c) $S \times T = T \times S$;
- (d) $R \cap (S \cap T) = (R \cap S) \cap T$;
- (e) $-(S \cap T) = -S \cap -T$;
- (f) $-(S \cup T) = -S \cap -T$;
- (g) $R \cap (S - T) = (R \cap S) - (R \cap T)$;
- (h) $R \cup (S - T) = (R \cup S) - (R \cup T)$

Exercise 4: If $A = \{1, 2, 3\}$ and $B = \{2, 4\}$ write down:

- (a) $A \cap B$;
- (b) $A \cup B$;
- (c) $A \times B$.

Exercise 5:

If $A = \{1, 2, 3\}$, $B = \{0, 1\}$ and $C = \{0, 2, 4\}$ find:

- (a) $A \cap (B \cup C)$;
- (b) $B \times C$.

Exercise 6: Let A, B be the following binary languages (sets of binary strings):

$A = \{1, 10, 101\}$, $B = \{1, 11, 011\}$.

Find each of the following sets:

- (a) $A \cap B$;
- (b) $A \cup B$;
- (c) $A - B$;
- (d) $A \times B$;
- (e) AB .

Exercise 7: Let R be the relation on the set of real numbers defined by xRy if $y = xq$ for some positive rational number q . Prove that R is an equivalence relation. Find an equivalence class with a finite number of elements.

Exercise 8: Let \approx be the relation on the set \mathbb{R} where $x \approx y$ means ' $x^m = y^n$ for some positive integers m, n '. Prove that \approx is an equivalence relation. Which equivalence classes are finite?

Exercise 9: The relation R on the set $\{2, 3, 4, 8, 15\}$ defined by $xRy \leftrightarrow$ ' $x + y$ is not prime' is an equivalence relation. Find the equivalence classes.

Exercise 10: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

- (a) R on the set \mathbb{N} where $xRy \leftrightarrow x \geq y$;
- (b) R on the set \mathbb{N} where $xRy \leftrightarrow x^2 = y^2$;
- (c) R on the set \mathbb{R}^+ where $xRy \leftrightarrow \log(y/x)$ is an integer.

Exercise 11: Let R be the relation on the set of integers defined by xRy if $x + y$ is even. Prove that R is an equivalence relation.

Exercise 12: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

- (a) R on the set P of people where $xRy \leftrightarrow x$ is the brother of y ;
- (b) R on the set \mathbb{Q} where $xRy \leftrightarrow y = x \cdot 2^n$ for some $n \in \mathbb{Z}$.

Exercise 13:

Define the relation R on the set $S = \{1, 2, 3, \dots, 100\}$ by:

$$aRb \leftrightarrow a \mid 2^r b \text{ for some } r \in \mathbb{N} \text{ and}$$

$$b \mid 2^s a \text{ for some } s \in \mathbb{N}.$$

- (a) Prove that R is an equivalence relation.
- (b) Find the number of equivalence classes.

Exercise 14: Let R be the relation on \mathbb{N} defined by xRy if $3x + 4y$ is a multiple of 7. Prove that R is an equivalence relation.

Exercise 15: Let $\alpha = \text{'string'}$ and $\beta = \text{'length'}$.

Find (a) $\alpha\beta$; (b) $\beta\alpha$; (c) $\beta\lambda$;

(d) strings γ, δ such that $|\gamma| = 2$ and $\gamma\delta = \alpha$.

SOLUTIONS FOR CHAPTER 4

Exercise 1: (a) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9; (b) 3, 5, 11; (c) 1, 9; (d) 0, 2, 4, 6, 7, 8; (e) 1, 2, 3, 5, 7, 9, 11; (f) (3,1), (3,9), (5,1), (5,9), (11,1), (11,9).

Exercise 2: (a)(e) S; (b)(c)(d) \emptyset .

Exercise 3: (a), (d), (f), (g).

Exercise 4: (a) $A \cap B = \{2\}$; (b) $A \cup B = \{1, 2, 3, 4\}$; (c) $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$.

Exercise 5: (a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{0, 1, 2, 4\} = \{1, 2\}$;
(b) $B \times C = \{(0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4)\}$.

Exercise 6: (a) $\{1\}$; (b) $\{1, 10, 11, 011, 101\}$;
(c) $\{10, 101\}$;
(d) $\{(1, 1), (1, 11), (1, 011), (10, 1), (10, 11), (10, 011), (101, 1), (101, 11), (101, 011)\}$;
(e) $\{11, 111, 1011, 101, 10011, 10111, 101011\}$.

Exercise 7: *Reflexive:* $x = x1$ and 1 is rational.

Symmetric: Suppose xRy .

Then $y = xq$ for some $q \in \mathbb{Q}^+$ and so $x = yq^{-1}$ and since $q^{-1} \in \mathbb{Q}^+$, yRx .

Transitive: Suppose xRy and yRz .

Then $y = xq$ and $z = yr$ for some $q, r \in \mathbb{Q}^+$.

Then $z = (xq)r = x(qr)$. Since $qr \in \mathbb{Q}^+$, xRz .
 $\{0\}$ is the only finite equivalence class.

Exercise 8: *Reflexive:* $x^1 = x^1$ so $x \approx x$ for all x .

Symmetric: Suppose $x \approx y$. Thus, $x^m = y^n$ for some positive integers m, n .

Then since $y^n = x^m$, $y \approx x$.

NOTE: The fact that the definition says ‘for some positive integers m, n ’ means that they don’t have to be the same ones each time.

We do not have to show that $x^m = y^n \rightarrow y^m = x^n$.

Transitive: Suppose $x \approx y$ and $y \approx z$. Thus $x^m = y^n$ and $y^s = z^t$ for some positive integers m, n, s, t .

Hence $x^{ms} = (x^m)^s = (y^n)^s = (y^s)^n = z^{nt}$ and so $x \approx z$.

The only finite equivalence classes are $\{0\}$ and $\{1, -1\}$.

All others are infinite.

eg $[2] = \{\pm 2^n \mid n \in \mathbb{Z}\}$.

Exercise 9: $\{2, 4, 8\}$, $\{3, 15\}$.

Exercise 10: (a) RT; (b) RST; (c) none; (d) RST.
(b)(d) are equivalence relations.

Exercise 11:

Reflexive: $x + x = 2x$ is even for all integers x ;

Symmetric: if $x + y$ is even then $y + x$ is even;

Transitive: Suppose $x + y$ and $y + z$ are even.

Then $(x + y) + (y + z) = x + z + 2y$ is even, and so $x + z$ is even.

Exercise 12: (a) Symmetric

Not Reflexive since I am not my own brother.

Not Transitive: take $x = z = \text{me}$ and $y = \text{my brother}$.

(b) Reflexive, Symmetric and Transitive (i.e. an equivalence relation).

Exercise 13:

(a) *Reflexive:* Let $a \in S$. Then since $a \mid 2a$, aRa .

Symmetric: R is clearly symmetric as defined.

Transitive: Suppose aRb and bRc .

Then $a \mid 2^r b$, $b \mid 2^s a$, $b \mid 2^u c$, $c \mid 2^v b$ for some natural numbers. Hence $a \mid 2^{r+u} c$ and $c \mid 2^{v+s} a$.

Since $r + u$ and $v + s$ are both natural numbers it follows that aRc .

Thus we've shown that R is an equivalence relation.

(b) Two numbers are equivalent under R if they are equal, give or take some factors of 2. Thus each equivalence class contains exactly one odd number and so the number of equivalence classes is exactly 50.

Exercise 14: *Reflexive:* Let $x \in \mathbb{N}$. Then $3x + 4x = 7x$ is a multiple of 7 and so xRx .

Symmetric: Suppose xRy . Then $7 \mid 3x + 4y$.

Thus $7 \mid 7(x + y) - (3x + 4y)$, that is, $7 \mid 4x + 3y$.

Hence yRx .

Transitive:

Suppose xRy and yRz . Then $7 \mid 3x + 4y$ and $7 \mid 3y + 4z$.

Hence $7 \mid 3x + 7y + 4z$ and so $7 \mid 3x + 4z$. So xRz .

Exercise 15: (a) ‘stringlength’; (b) ‘lengthstring’;
(c) ‘length’; (d) $\gamma = \text{‘st’}$, $\beta = \text{‘ring’}$.