5. COUNTING

§5.1. The Art of Counting

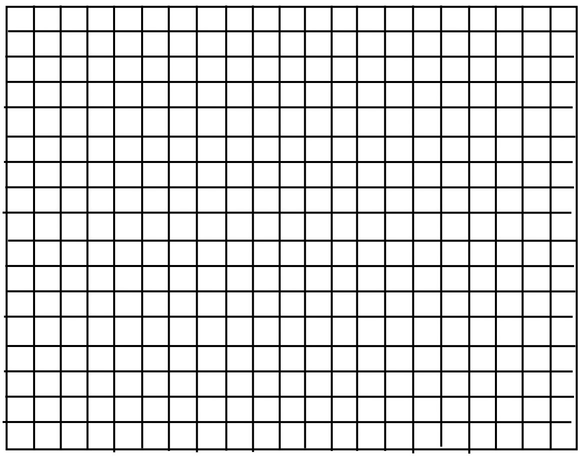
Counting was the first mathematical skill you ever mastered. Many mathematical theorems are based on counting. For example two sets can be proved to be different by simply showing that they have different sizes. And a subset can be shown to be the whole set by showing that they have the same size, though this only works if the sets are finite.

Counting is important in computing science. The complexity of an algorithm can be determined by counting the number of steps or amount of memory it takes. By examining how these numbers grow with the size of the task we can determine how useful the algorithm is. If we have an algorithm where the number of steps grows exponentially with the size of the problem it is of little use in practice.

It's quite elementary to count a finite set if the elements are listed. Counting an infinite set is quite another matter. You might think that it is sufficient to say that a set is infinite, or that its number of elements is ∞. But there are many infinite numbers – infinitely many in fact. Although the sets of integers, rational numbers and real numbers are all infinite, the number of elements in $\mathbb{Z}$ and $\mathbb{Q}$ is the same while the number of elements of $\mathbb{R}$ is bigger. Yes, a bigger infinite number!

But here, in this chapter, we'll be restricting our attention to finite sets. So what is there to say? A finite set can be very large and it might be too tedious to count each element individually. If the set has some structure to it we can usually count its elements using some mathematical calculation. A simple example is where we have a table with m rows and n columns. The number of cells is simply mn. There's no need to count each of them.

Example 1: How many small squares are there in the following diagram?



Solution: There are 17 rows and 21 columns so there are $17 \times 21 = 357$ little squares.

We denote the number of elements of a set S by **#S** (sometimes it is denoted by |S|.)
If S is finite, #S is a natural number. Note that $\#\varnothing = 0$, where $\varnothing$ denotes the empty set.
If S is a subset of a finite set T then #(S – T) = #S – #T. So if there are 150 students enrolled in a course and 60 are female then 90 are male.

Example 2: How many of the numbers from 1 to 20 are not perfect squares.
Solution: The perfect squares are 1, 4, 9 and 16 so there are 4 perfect squares.
This leaves 20 – 4 = 16 numbers that are not perfect squares.

Theorem 1: $\#(S \cup T) = \#S + \#T - \#(S \cap T)$.
Proof: If we simply added the sizes of the two sets we would have double-counted those that are in both. So we need to subtract off $\#(S \cap T)$. ✋☺
NOTE: This is a very simple case of the Inclusion-Exclusion Principle that we'll study later.

Theorem 2: $\#(S \times T) = \#S \times \#T$, where $S \times T$ is the set of ordered pairs (s, t), with $s \in S$ and $t \in T$.
Proof: This is equivalent to counting the individual cells in an array with *m* rows and *n* columns. The number is *mn*. ✋☺

Theorem 3: $\#\wp S = 2^{\#S}$.
Proof: In other words, if a set has n elements it has 2^n subsets.
For each of the n elements there's a choice of including it or of excluding it, that is 2 choices. Making n independent such choices there are 2^n combinations of these choices. But each combination of choices corresponds to a subset and vice versa. ✋☺

Example 3:
How many subsets are there of the set {1, 2, 3}.
Solution: The answer is $2^3 = 8$. To help explain the above proof we'll list all 8 subsets together with the corresponding choice combinations (N = no, Y = yes).

subset	**choices**
∅	NNN
{1}	YNN
{2}	NYN
{3}	NNY
{1, 2}	YYN
{1, 3}	YNY
{2, 3}	NYY
{1, 2, 3}	YYY

§5.2. The Number of Choices

We define $\boldsymbol{n!} = n(n - 1)(n - 2) \ldots 3.2.1$ This is called **factorial *n***. It is the number of arrangements of n things.

Example 4: The number of ways of arranging the four numbers 1, 2, 3, 4 is 4! = 4.3.2.1 = 24.
These arrangements are:

1234, 1243, 1324, 1342, 1423, 1432,
2134, 2143, 2314, 2341, 2413, 2431,
3124, 3142, 3214, 3241, 3412, 3421,
4123, 4132, 4213, 4231, 4312, 4321.

Notice that I've listed these arrangements systematically to make counting easier. In each row we have a different number chosen as first and then the remaining 3 numbers are arranged in all 3! = 6 ways.

We define $^{n}\mathbf{P}_{r}$ to be the number of ways of choosing *r* things from *n* if the order is important.

Example 5: In a competition they award two prizes, First and Second. If there are 5 contestants, Alice, Bob, Carl, Daisy and Elise how many possible results are there?
Solution: Here, the order is important. Alice first and Bob second is a different outcome to Bob first and Alice second.

There are 5 possibilities for First Prize, and for each of these there are 4 possibilities for Second. So, in all there are $5 \times 4 = 20$ possible outcomes. If we represent each contestant by their initial the 20 outcomes are:

AB, AC, AD, AE,
BA, BC, BD, BE,
CA, CB, CD, CE,
DA, DB, DC, DE,
EA, EB, EC, ED.

Again, you'll notice that we have listed the choices systematically. By listing things in some systematic order we can ensure that our count is exact. If we'd listed them in some haphazard way we're likely to have repeated one or left some out.

We define $^{n}C_{r}$ to be the number of ways of choosing *r* things from *n* if the order is not important.

Example 6: There are 9 players in a tennis club that plays on Thursday nights. Each week they play three sets, with different players. The captain makes up a roster in which he chooses the 4 players for each set. He wants to have each group of 4 to play exactly once during the season. The order doesn't matter, because it's left to the 4 players as to who plays with whom. How many weeks will it take for all the possible groups of 4 to play a set?
Solution: The number of sets will be $^{9}C_{4}$. If the order mattered he would have 9.8.7.6 choices, but each group of 4 would get counted 4! = 24 times so we would need to divide $^{9}C_{4}$ by 4!.

Hence the number of choices of 4 will be $\frac{9.8.7.6}{4.3.2.1} = 126$. With 3 sets per week it will take 42 weeks.

A separate question is how should the 126 choices be distributed over the 42 nights. If they were scheduled at random some players wouldn't get a game on certain nights. The decision as to the best allocation would depend on whether you wanted to maximise the nights a player need not come or whether you wanted to ensure that each player was involved in at least one set each evening. We won't go into that.

The more usual notation for ${}^nC_r$ is $\binom{\boldsymbol{n}}{\boldsymbol{r}}$. It is called a **binomial coefficient**. This is because these numbers arise as coefficients in the expansion of $(x + y)^n$. We'll discuss this later when we come to the Binomial Theorem. By the way, 'binomial' simply means 'two terms', such as the x and y in $x + y$.

Theorem 4: ${}^nP_r = \frac{n!}{(n-r)!}$ and $\binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Proof: We first find a formula for ${}^nP_r$. Clearly, if we have to choose r things from n and the order is important, the number of choices is:

$$n(n-1)(n-2) \ldots (n-r+1).$$

The last factor is $n - r + 1$ because there are r factors.

This can be written neatly as ${}^nP_r = \dfrac{n!}{(n-r)!}$.

But, if order is not important, we must divide by $r!$, representing the number of ways any particular choice can be rearranged. ✋☺

Example 7: $\dbinom{9}{4} = \dfrac{9!}{4!5!} = \dfrac{9.8.7.6.5.4.3.2.1}{4.3.2.1.5.4.3.2.1}$

$$= \frac{9.8.7.6}{4.3.2.1} = 126.$$

It's not necessary to include factors of 1, but it's a good idea to do so because we can then see that the number of factors is the same in both the numerator and denominator before we start to cancel.

Although $\dfrac{n!}{r!(n-r)!}$ is a very neat formula for $\dbinom{n}{r}$ it involves a lot of unnecessary factors that get cancelled. An easier way to calculate $\dbinom{n}{r}$ is to start with n on the top of the fraction and r on the bottom and run each of them down by 1 until you reach 1 on the bottom. The numerator and denominator will have the same number of factors.

Example 7 (again):

$\dbinom{9}{4} = \dfrac{9.8.7.6}{4.3.2.1}$. This involves much less unnecessary cancelling.

The calculation can be made even simpler by using the following fact.

Theorem 5: $\binom{n}{r} = \binom{n}{n-r}$.

Proof: An algebraic proof, using factorials, is not too difficult. But by far the easiest proof is to recognise that when we choose r things from n we are in effect choosing the $n - r$ things that we reject. The number of choices is the same. ✋☺

Example 8: $\binom{100}{98} = \binom{100}{2} = \frac{100.99}{2.1} = 4950.$

This is so much easier than $\frac{100.99.98. \ldots .3}{98.97.96. \ldots .1}$

Theorem 6: $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$.

Proof: The left hand side is the number of subsets of size r of the set $\{0, 1, 2, \ldots, n\}$. We can separate these into those subsets that contain 0 and those that don't.

Those that contain 0 will consist of 0 plus a further $r - 1$ elements from the set $\{1, 2, \ldots, n\}$.

Clearly there will be $\binom{n}{r-1}$ of these.

The subsets that don't contain 0 will be a subset of size r from the set $\{1, 2, \ldots, n\}$.

There will be $\binom{n}{r}$ of these.

Hence $\binom{n+1}{r} = \binom{n}{r-1} + \binom{n}{r}$. ✋☺

Theorem 7: $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} \ldots + \binom{n}{n-1} + \binom{n}{n} = 2^n$.

Proof: The number of subsets of $\{1, 2, 3, \ldots, n\}$ is 2^n and $\binom{n}{r}$ is the number of subsets of size r. The theorem simply counts the subsets by counting the numbers of each size. ✋☺

Theorem 8: The number of ways of choosing r things from n (where $r \le n$) depends on whether we allow repetitions and whether we want to distinguish between different orderings.

The number of choices is given in the following table.

	ordered choices	**unordered choices**
no repetitions allowed	$\frac{n!}{(n-r)!}$	$\binom{n}{r}$
repetitions allowed	n^r	$\binom{n+r-1}{r-1}$

We've seen the case of no repetitions allowed so let's consider the case of choices with repetitions.

Example 9: A bag contains 5 tickets in a raffle. Three prizes are awarded, first second and third. For each prize a ticket is drawn and replaced. So that one ticket could win more than one prize. How many possible outcomes will there be?

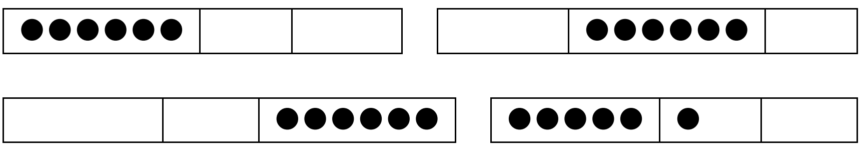
Solution: Suppose the tickets are numbered 1, 2, 3, 4 and 5. The choices are 111, 112, 113, 114, 115, 211, 212, …. , 555. There are 5 possibilities for the 1st choice. For each of these there are 5 possibilities for the 2nd choice, giving $5 \times 5 = 25$ possibilities for the 1st two choices. For each of these there are 5 possibilities for the 3rd choice, giving $5^3 = 125$ choices altogether.

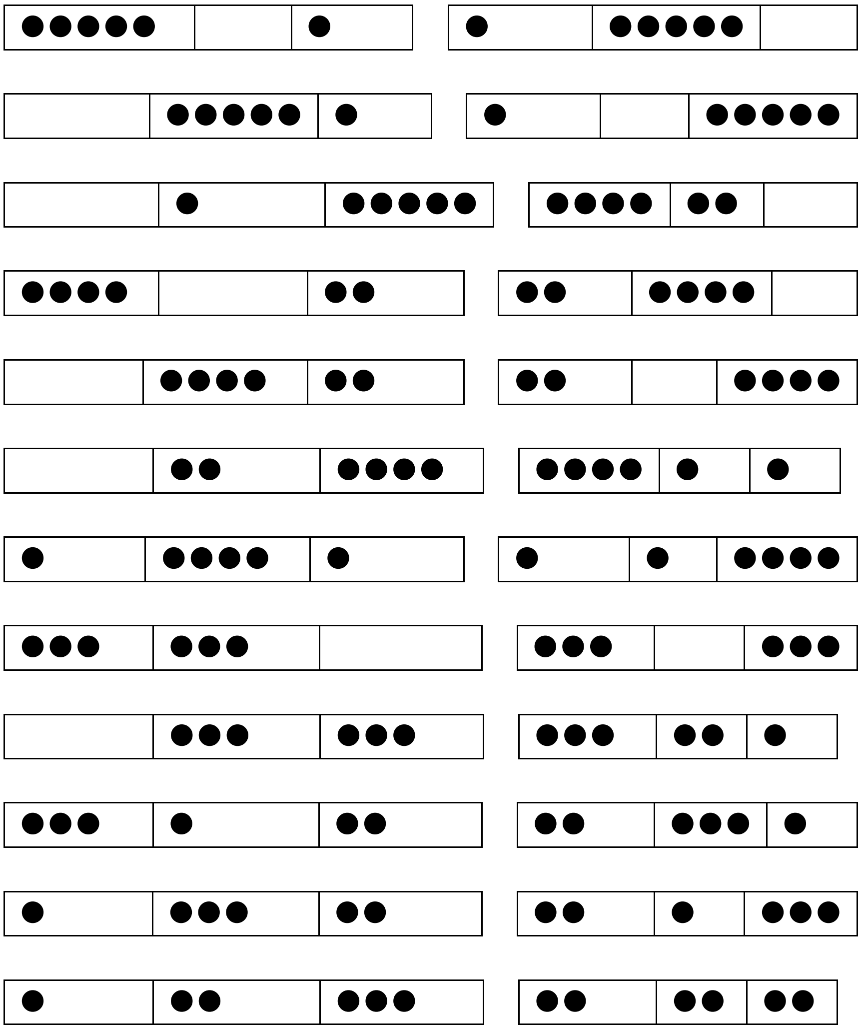
So, in the general case where repetitions are allowed and order has to be considered, the number of choices of *r* from n is n^r. The most difficult case is where the order is where repetitions are allowed, but we have unordered choices. This is the number of ways of putting *n* things into *r* boxes.

Example 10: I have six identical balls and want to put them into 3 boxes, a red box, a blue box and a yellow box. In how many ways can I do this? Each box is big enough to accommodate all six items, so one possibility is to put all 6 in one box, leaving the other two boxes empty.

Solution:

If we represent a ball by ●, the possibilities are:





In all there are 28 possibilities.

We could have counted these, without actually producing all the possibilities, as follows. We consider the partitions, without bothering about which box is which, and then count the possibilities for that partition.

600: There are **3** boxes in which the 6 balls can go.
510: There are **3** boxes in which the 5 can go, and for each of these, there are 2 boxes for the 1, making **6** in all.
420: Again there are **6** possibilities.
411: Just **3** possibilities.
330: Again **3** possibilities.
321: Here, because the numbers are all different, there are **6** possibilities.
222: Just **1** possibility here.
3 + 6 + 6 + 3 + 3 + 6 + 1 = 28.

But Theorem 8 gives the quickest method for calculating the number of possibilities. Here $n = 6$ and $r = 3$.

$$\binom{n+r-1}{r-1} = \binom{8}{2} = \frac{8.7}{2.1} = 28.$$

Proof of Theorem 8: Repetitions allowed/unordered choices:
We can write each choice as a sequence of n ●'s, representing the things, and $r - 1$ |'s, representing the divisions between boxes. In the above example we can denote

●	●●●	●

by ●|●●●●|●.

Altogether we now have $n + r - 1$ symbols. A partition is determined once we choose where to place the dividing lines. From the $n + r - 1$ we choose $r - 1$ positions in which to place the dividers. All the other positions will be occupied by ●'s. Hence the number of such choices is:

$$\binom{n+r-1}{r-1}.$$

Example 11: In how many ways can you write 6 as a sum of three natural numbers, counting sums separately if the terms are rearranged?
(Include 6 = 6 + 0 + 0, 6 = 0 + 6 + 0 and 6 = 0 + 0 + 6 as three of these possibilities.)
Solution: This is an equivalent problem to the one above and so there are 28 possibilities.

If we wish to consider different arrangements of the same terms as being the same then the number of possibilities is reduced to 7. They are:

6	5 + 1	4 + 2	3 + 3	4 + 1 + 1	3 + 2 + 1	2 + 2 + 2

If we remove the restriction on the number of terms we add an extra 4 possibilities:

2 + 2 + 1 + 1	2 + 1 + 1 + 1

2 + 1 + 1 + 1 + 1	1 + 1 + 1 + 1 + 1 + 1

There is no simple formula for these scenarios.

§5.3. The Binomial Theorem

We remember from school that $(a + b)^2 = a^2 + 2ab + b^2$. The Binomial Theorem generalizes this to $(a + b)^n$.

Theorem 9: $(a + b)^n = a^n + \binom{n}{1} a^{n-1}b + \binom{n}{2} a^{n-2}b^2 + \ldots$

$$\ldots + \binom{n}{r} a^{n-r}b^r + \ldots + \binom{n}{1} ab^{n-1} + b^n.$$

Proof: When we expand $(a + b)(a + b) \ldots (a + b)$ we get a sum of terms where each one consists of a product of n factors. Each such factor is chosen from one of the n factors of $(a + b)^n$.

A typical term will consist of r a's and $n - r$ b's in some order. But since the order of the factors is irrelevant we write the term as $a^r b^{n-r}$. We'll have as many terms that equate to $a^r b^{n-r}$ as there are different ways we can choose the r a's and the $n - r$ b's. The number of terms that equate to $a^r b^{n-r}$ will be the number of ways of choosing the r factors from $(a + b)^n$ from which we choose an a and this is $\binom{n}{r}$. This is then the coefficient of $a^r b^{n-r}$ in the expansion of $(a + b)^n$.

Alternative Proof: This can also be proved by induction. It is clearly true for n = 1.
Suppose it is true for n.

$(a + b)^{n+1} = (a + b)^n(a + b)$.
A term that results in $a^r b^{n+1-r}$ will either come as $a^{r-1}b^{n+1-r}$ times a or as $a^r b^{n-r}$ times b.

There are $\binom{n}{r-1}$ and $\binom{n}{r}$ respectively.

So the coefficient of $a^r b^{n+1-r}$ in $(a + b)^{n+1}$ will be:

$\binom{n}{r-1} + \binom{n}{r} = \binom{n+1}{r}$, by Theorem 6.

Example 12:
$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$.
$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$.
$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 4ab^4 + b^5$.

Notice the pattern that these expansions follow. We start with $a^n = a^n b^0$. Then in each successive term the power of a goes down by 1 and the power of b goes up by 1. The sum of these powers is always n. These coefficients are the binomial coefficients.

These binomial coefficients follow a very nice pattern. Suppose we write them out without the a's and b's in a triangular shape.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
....

This triangle is called **Pascal's Triangle** after the mathematician Blaise Pascal. The $(n + 1)^{st}$ row gives the coefficients in the expansion of $(a + b)^n$.

Notice the pattern. The first thing you'll notice are the left and right edges which are all 1's. Then you'll notice the left-right symmetry as observed in Theorem 5. In each row the numbers grow, reach a maximum in the middle, and then decrease again. A little less obvious is the fact that each number, apart from the 1's, is the sum of the two numbers immediately above it. For example, 10 = 4 + 6.

4   6
10

This fact is simply a consequence of Theorem 6. It enables the successive rows of the triangle to be generated very easily. For example we can continue the triangle as follows.

…………..
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
………………………………

§5.4. Counting Partitions and Equivalence Relations

A **partition** on a set S is a set of subsets $S_1, S_2, \ldots, S_n$ such that each pair of distinct subsets is disjoint and their union is S. Every element of S will therefore belong to exactly one of these subsets.

S_1	S_2	S_3	....	S_n

Transitive relations are rather difficult to count. But we can count equivalence relations using equivalence classes. Every equivalence relation on S corresponds to a partition of S into equivalence classes and every partition of S corresponds to an equivalence relation. So we simply need to count the partitions.

Example 13: The equivalence relation:
{(1, 1), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)} corresponds to the partition {{1, 4}, (2}, {3}}.

Example 14: How many partitions are there on the set {1, 2, 3, 4}?
Solution: We begin by enumerating the types of partition. The above partition consists of a pair and two singles. We can represent this type by (××)(×)(×). With 4 elements the possible types of partition are:
(××××), (×××)(×), (××)(××), (××)(×)(×), (×)(×)(×)(×).
Now we have to count the numbers of partitions of each type. For (××××) there's only one partition – all in together. For (×××)(×) there are 4 choices for the singleton. Having chosen which one goes by itself there's no further choice – all the other three go in together. For the (××)(×)(×) type we have ${}^4C_2$ = 6 ways of choosing the

two singletons and so 6 partitions of that type. And there is only one partition of the type (×)(×)(×)(×).

The partitions of the type (××)(××) are a little more complicated to count. To start with we have $^4C_2 = 6$ choices for the first pair, with no further choice for the second. But because the two pairs have the same size they're interchangeable and so each partition would have been counted twice. For example {1, 3}, {2, 4}} and{{2, 4}, {1, 3}} are the same partition. So we have just 3 partitions of this type.

We can summarise this as follows:

Type	**Number**
(××××)	1
(×××)(×)	4
(××)(××)	3
(×××)(×)(×)	6
(×)(×)(×)(×)	1
TOTAL	**15**

Thus there are exactly 15 equivalence relations on this set.

§5.5. The Inclusion-Exclusion Principle

Suppose we have a collection of subsets of some larger set. For example if we number the courses at some educational establishment 1, 2, … , *n* we could define S_r

to be the set of students enrolled in course n. How many students are enrolled in at least one course? The answer will be somewhat less than $\#S_1 + \#S_2 + \ldots \#S_n$ because most students will be enrolled in several course and will be counted many times. So we must subtract the number who are enrolled in both S_r and S_t for various values of r and t.

But then we may have over compensated. Students enrolled in three courses will be counted 3 times in the first sum but, their number will be subtracted off 3 times because they're enrolled in 3 pairs of courses. They must be added back.

Theorem 10 (INCLUSION-EXCLUSION):

$$\begin{aligned}\#(S_1 \cup S_2 \cup \ldots \cup S_n) = {} & \#S_1 + \#S_2 + \ldots + S_n \\ & - \#(S_1 \cap S_2) - \ldots - \#(S_{n-1} \cap S_n) \\ & + \#(S_1 \cap S_2 \cap S_3) + \ldots \\ & - \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ & + (-1)^n \#(S_1 \cap S_2 \cap \ldots \cap S_n).\end{aligned}$$

Proof: The above discussion should be sufficient proof. However a more formal proof can be given by mathematical induction.

Example 15: Suppose the enrolments in 4 courses are as follows:

MATHS	99
STATS	79

COMP	84
PHYSICS	93
MATHS & STATS	44
MATHS & COMP	51
MATHS & PHYSICS	60
STATS & COMP	37
STATS & PHYSICS	47
COMP & PHYSICS	56
MATHS, STATS & COMP	13
MATHS, STATS & PHYSICS	18
MATHS, COMP & PHYSICS	33
STATS, COMP & PHYSICS	23
all 4 courses	3

How many students are enrolled in at least one course? The number is:

$$99 + 79 + 84 + 93 - 44 - 51 - 60 - 37 - 47 - 56 + 13 + 18 + 33 + 23 - 3 = 144.$$

§5.6. Number of Non-Negative Solutions to $x_1 + x_2 + \ldots + x_r = n$.

Example 16: How many non-negative solutions are there to the equation $x + y + z = 5$?

Solution: The solutions are:

$0 + 0 + 5 = 5$

$0 + 1 + 4 = 5$

$0 + 2 + 3 = 5$

$0 + 3 + 2 = 5$

$0 + 4 + 1 = 5$
$0 + 5 + 0 = 5$
$1 + 0 + 4 = 5$
$1 + 1 + 3 = 5$
$1 + 2 + 2 = 5$
$1 + 3 + 1 = 5$
$1 + 4 + 0 = 5$
$2 + 0 + 3 = 5$
$2 + 1 + 2 = 5$
$2 + 2 + 1 = 5$
$2 + 3 + 0 = 5$
$3 + 0 + 2 = 5$
$3 + 1 + 1 = 5$
$3 + 2 + 0 = 5$
$4 + 0 + 1 = 5$
$4 + 1 + 0 = 5$
$5 + 0 + 0 = 5.$

Counting the solutions we see that there are 21 of them. It should be obvious that generating all the solutions is a very tedious of finding the number of solutions. Surely there's a more efficient technique.

Theorem 11: The number of non-negative integer solutions to the equation:

$$x_1 + x_2 + \ldots + x_r = n \text{ is } \binom{n+r-1}{r-1}.$$

Proof: Each such solution corresponds to a partition of n things into r compartments. The number of solutions is $\binom{n+r-1}{r-1}$ by Theorem 8.

Example 16 (continued):
Using Theorem 11 we can quickly find the answer to Example 16.
The number of solutions is $\binom{3+5-1}{2} = \binom{7}{2} = 21$.

Now suppose that, instead of all non-negative solutions to the equation $x + y + z = 5$ we want the number of solutions where $x \geq 2$. Having taken the trouble listing all the solutions it's a simple exercise to count those where $x \geq 2$. The number is 10. However counting would not be feasible if the total was 55 instead of 5. There would be too many solutions.

If $x \geq 2$ then $u = x - 2 \geq 0$ and $u + y + z = 3$.

The non-negative solutions to $x + y + z = 5$ where $x \geq 2$ are in 1-1 correspondence with the non-negative solutions to $u + y + z = 3$.
By Theorem 11 this number is $\binom{3+3-1}{2} = \binom{5}{2} = 10$.

Theorem 12: The number of non-negative solutions of

$$x_1 + x_2 + \ldots + x_r = n \text{ with each } x_i \geq k$$

is the number of non-negative solutions of:

$x_1 + x_2 + \ldots + x_r = n - rk$ which is $\binom{n + r - rk - 1}{n - 1}$.

Proof: The solutions to $x_1 + x_2 + \ldots + x_r = n$ with each $x_i \geq k$ correspond to the solutions to:

$$x_1 + x_2 + \ldots + x_r = n - rk$$

with only the non-negativity condition on all variables (by replacing each x_i by $x_i - k$).

NOTE: To incorporate the condition $x_i \geq k$ simply reduce the top parameter in the binomial coefficient by k for each such condition.

Example 17: How many non-negative integer solutions are there to the equation $x + y + z = 5$ for which $x \geq 2$ and $y \geq 1$?

Solution: Let $u = x - 2$ and $v = y - 1$. Then $u + v + z = 2$. The number of solutions for x, y and z, with the stated conditions, is the same as the number of solutions for u, v and z with no extra conditions. This is $\binom{4}{2} = 6$.

Example 18: How many non-negative integer solutions are there to the equation $x + y + z = 5$ for which $x \leq 2$?

Solution: We calculate the number of solutions where

$x \geq 3$ and subtract from the total number of solutions. The total number of solutions is $\binom{7}{2} = 21$. The number of solutions where $x \geq 3$ is $\binom{4}{2} = 6$, so the number of solutions where $x \leq 2$ is $21 - 6 = 15$.

Example 19: How many non-negative integer solutions are there to the equation $x + y + z = 5$ for which $x \leq 2$ and $y \leq 1$?

Solution: There are $\binom{7}{2}$ solutions altogether. We treat the two conditions separately. Let S_1 denote the set of solutions where $x \geq 3$ and let S_2 denote the set of solutions where $y \geq 2$. The solutions to be deleted are those in $S_1 \cup S_2$. So the number of solutions satisfying the given inequalities is $\binom{7}{2} - \#(S_1 \cup S_2)$.

$$\begin{aligned} \text{Now } \#(S_1 \cup S_2) &= \#S_1 + \#S_2 - \#(S_1 \cap S_2) \\ &= \binom{4}{2} + \binom{5}{2} - \binom{2}{2} \\ &= 6 + 10 - 1 = 15. \end{aligned}$$

Hence the required number of solutions is $21 - 15 = 6$.

Example 20: How many non-negative integer solutions are there to the equation
$x_1 + x_2 + x_3 + x_4 = 18$ for which each $x_i \leq 5$?

Solution: There are $\binom{21}{3}$ solutions altogether.

Let S_i denote the set of solutions for which $x_i \geq 6$.

Then for each i, $\#(S_i) = \binom{15}{3}$.

For each distinct i, j $\#(S_i \cap S_j) = \binom{9}{3}$ and for each distinct i, j, k $\#(S_i \cap S_j \cap S_k) = \binom{3}{3}$.

Finally $\#(S_1 \cap S_2 \cap S_3 \cap S_4) = 0$ because if each x_i is at least 6 their sum would be at least 24.

Now there are $\binom{4}{2}$ pairs of distinct i, j and $\binom{4}{3}$ distinct triples. Hence the required number of solutions is:

$$\binom{21}{3} - \binom{4}{1}\binom{15}{3} + \binom{4}{2}\binom{9}{3} - \binom{4}{3}\binom{3}{3} + 0$$

$$= 1330 - 4.455 + 6.84 - 4.1 + 0 = 10.$$

§5.7. Counting Arrangements of Symbols

Example 21: How many arrangements are there of the letters AAABCC?

Solution: There are 6 letters, and if all of them were different there would be 6! = 720 different arrangements. But the 3A's can be permuted in 3! ways and the 2C's can be permuted in 2! ways. Such permutations result in the same arrangement and must be counted just once. We must therefore divide the 6! by 3! and by 2!.

The number of arrangements is therefore $\frac{6!}{3!2!} = 60$.

Example 22: How many arrangements are there of the letters A, D, O, R, W, X, Z that include the string WORD.
Solution: Imagine the letters to be written on cards. Because the strings must contain WORD we write these four letters on a single card as WORD. The remaining 3 letters are each written on a separate card. So we have 4 cards that we can permute in any of the 4! arrangements, and all of them will contain the string WORD simply because they occur on the same card. So there are 4! = 24 such arrangements.

Example 23: How many arrangements are there of the letters A, D, O, R, W, X, Z that don't include the string WORD.
Solution: There are 7 letters altogether and so 7! = 5040 arrangements. Of these 24 will contain the string WORD. Hence there are 5040 – 24 = 5016 arrangements that don't include the string WORD.

Example 24: The 26 letters A-Z are each written on a card. In how many ways can these cards be arranged so that none of the ten words ONE, TWO, … TEN occur.
Solution: The strings to be excluded are: ONE, TWO, THREE, FOUR, FIVE, SIX, SEVEN, EIGHT, NINE and TEN.

Of these THREE, SEVEN and NINE will be excluded automatically because they involve repeated letters.

Let S_i denote the number of arrangements that *do* include the word for the number *i*.

If that word contains repeated letters clearly $\#S_i = 0$. Otherwise, if that word has *n* letters then, replacing the cards with these letters by a single card containing the whole word, we have $26 - n + 1$ cards to be rearranged. In this case $\#S_i = (26 - n + 1)!$

For example, S_4 is the number of arrangements that do include FOUR. Replacing the four cards with the letters F, O, U, R by a single card containing the word FOUR we have 23 cards to be arranged in all 23! ways.

The sizes of the S_i's are as follows:

set	S_1	S_2	S_3	S_4	S_5	S_6	S_7	S_8	S_9	S_{10}
length	3	3	5	4	4	3	5	5	4	3
size	24!	24!	0	23!	23!	24!	0	22!	0	24!

If a string contains both ONE and TWO they must occur together as TWONE so the arrangements in $S_1 \cap S_2$ are those that contain TWONE. And $S_1 \cap S_4 = \varnothing$ because the O in FOUR can't be the O in ONE.

In the following table we consider $S_i \cap S_i$ for $i < j$. If the entry is $\varnothing$ this indicates that $S_i \cap S_j$ is empty as both words cannot occur. Otherwise there is listed one or two strings which must appear. Where only one is listed it is because the two words must run together sharing a letter.

	S_2	S_4	S_5
S_1	TWONE	∅	∅
S_2		∅	TWO, FIVE
S_4			∅
S_5			
S_6			
S_8			

	S_6	S_8	S_{10}
S_1	ONE, SIX	ONEIGHT	∅
S_2	TWO, SIX	EIGHTWO	∅
S_4	FOUR, SIX	FOUR, EIGHT	FOUR, TEN
S_5	∅	∅	∅
S_6		∅	SIX, TEN
S_8			∅

The number of arrangements in each $S_i \cap S_j$ is given as follows.

	S_2	S_4	S_5	S_6	S_8	S_{10}
S_1	22!	0	0	22!	20!	0
S_2		0	21!	22!	20!	0
S_4			0	21!	19!	21!
S_5				0	0	0
S_6					0	22!
S_8						0

Where there's just one composite string to be included, of length n, the number of cards being permuted is:

$$(26 - n + 1)! = (25 - n)!$$

Where there are two separate strings to be included, with n letters between them, the number of cards being permuted is:

$$(26 - n + 2)! = (24 - n)!$$

The set $S_i \cap S_j \cap S_k$ is only non-empty, for $i < j < k$, in the following cases. The required string(s) and the number of elements are given.

$S_1 \cap S_2 \cap S_6$	TWONE, SIX	20!
$S_1 \cap S_2 \cap S_8$	EIGHTWONE	16!
$S_4 \cap S_6 \cap S_{10}$	FOUR, SIX, TEN	19!

The only intersection of 4 or more subsets that's non-empty is $S_1 \cap S_2 \cap S_6 \cap S_8$.

$S_1 \cap S_2 \cap S_6 \cap S_8$	EIGHTWONE, SIX	16!

The number of arrangements that do not contain any of the strings ONE, … , TEN is therefore:

$$26! - 4.24! - 2.23! - 22! + 4.22! + 3.21! + 2.20! + 19! - 20! - 19! - 16! + 16!$$

$$= 26! - 4.24! - 2.23! + 3.22! + 3.21! + 20!$$

Using a high-precision calculator, we can evaluate this as:

$$20!(165765600 - 1020096 - 21252 + 1386 + 63 + 1)$$

$$= 400,761,491,194,106,764,001,280,000.$$