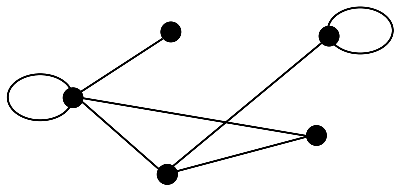
6. GRAPHS

§6.1. Graphs in Maths and Computing

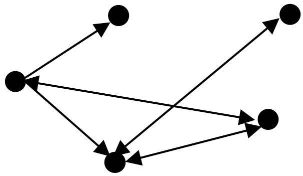
In mathematics the word 'graph' is used to describe a picture of a function $y = f(x)$ against two axes. But there's another sort of graph – one that is studied in Graph Theory. A graph is a collection of objects, called **vertices**, and connections between vertices, called **edges**.



We don't, however, allow multiple edges between the same pair of vertices, though some references do allow this.

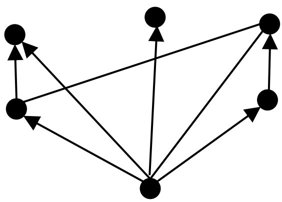
We allow **loops**, that is, edges from a vertex to itself, though in many applications our graphs will be **loop-free**, that is they'll have no loops.

In general the edges of a graph may have a direction. If so, we represent the edges in the picture of the graph as arrows. Such a graph is called a **directed graph**. For example friendship is often one way. Jack might nominate Joe as a friend but Joe might not consider Jack as a friend.



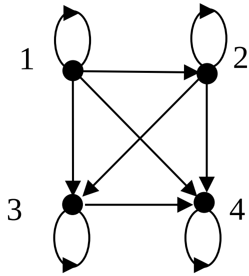
A directed graph is nothing more than a relation on the set of vertices. A reflexive relation will be represented by a graph with a loop at each vertex. A symmetric relation will be represented by a directed graph in which each arrow goes in both directions. In such cases we can remove the arrow heads and we have an **undirected graph**.

Example 1: For example, the relation of divisibility on the set {1, 2, 3, 4, 5, 6} can be represented by the graph:

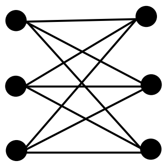


I have deliberately not labelled the vertices. See if you can work out which number each vertex represents,

Example 2: See if you can work out what well-known relation on {1, 2, 3, 4} is represented by the following directed graph:



We often represent vertices by points and edges by lines and when we draw a graph on a surface it can tell us something about the topology of the surface. For example we can prove that the following graph can't be drawn on a plane without some edges crossing. However it can be so drawn on a doughnut without the edges crossing.

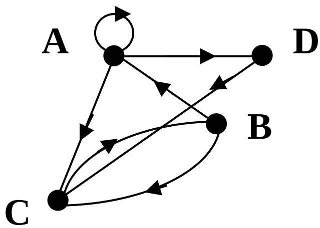


But very often graphs have no geometric or topological significance. For example the vertices might represent people and the edges might connect pairs of friends.

§6.2. Basic Definitions for Graphs

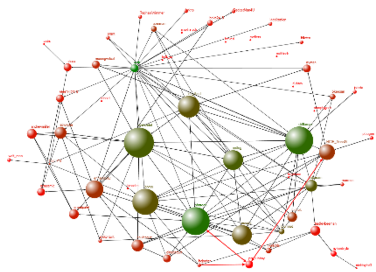
A **graph** is a set, X, of elements together with a relation on X. We often draw pictures of graphs, where the elements of X are represented by dots and the relation by a set of arrows connecting certain dots to others. The elements of X are called **vertices** and the arrows joining vertices related by the relation are called **edges**.

Example 3:
The following is a graph on the set {A, B, C, D}:
{(A, A), (A, C), (A, D), (B, A), (B, C), (C, B), (D, C)}.
This can be represented by the following diagram. There are 4 vertices and 7 edges.



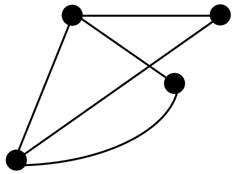
The edges of this graph can be listed very concisely as AA, AC, AD, BA, BC, DC.

A **loop** is an edge (V, V) from a vertex to itself.



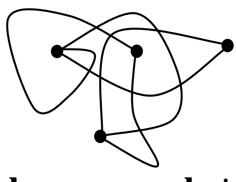
An **undirected graph** is one where the relation is symmetric – if U is related to V then V is related to U. So if there's an arrow in one direction there's always one in the opposite direction. In an undirected graph there's no need to use arrows since we know that the relation goes in both directions, so we simply use edges without arrows. A graph with arrows is called a **directed graph**.

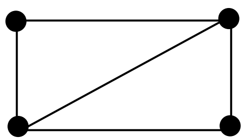
Example 4: The following is an undirected graph with no loops.



A graph is a combinatorial structure where the only consideration is which vertices are adjacent to which. When we draw a graph the positions of the points representing the vertices are arbitrary. So are the routes of the edges. The edges needn't be straight, they are allowed to cross over other edges, and they could even wind around in more complicated ways. However we usually draw a graph in such a way that it gives as simple a picture as possible.

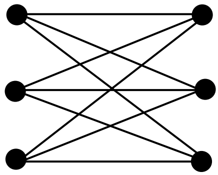
Example 5: The graph in example 4 could be redrawn as in the diagram on the left, but would look much better when drawn as the one on the right.





A **planar graph** is one that can be drawn on a plane without edges crossing. Clearly the graph in example 4 is planar.

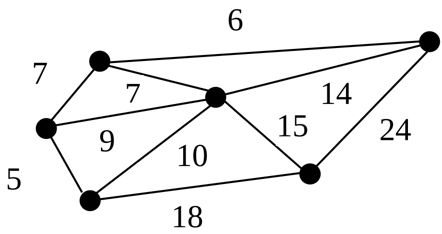
Example 6: The following is $\mathbf{K_{3,3}}$:



This graph was once featured in an Air New Zealand advertisement, where the 6 vertices consisted of the cities Brisbane, Sydney, Melbourne, Auckland, Wellington and Christchurch. The edges represented the trans-Tasman routes. This graph is not planar. You'll be easily convinced of this fact if you try to draw it without edges crossing, but that won't constitute a proof that it can't be done. However it can be drawn on a doughnut without edges crossing!

The question of whether certain graphs can be drawn on certain surfaces is an interesting question in Topology. Planarity is discussed at the end of this chapter. For other surfaces it is discussed in my notes on Topology.

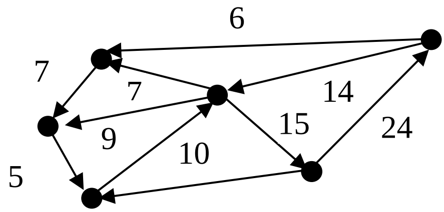
A graph (either directed or undirected) is called **weighted** if there is a number associated with each edge. A famous problem in computing science is the so-called Travelling Salesman Problem. Given a weighted undirected graph, of all the paths that starts at one vertex, visit every other vertex and return to the one where it started, find one for which the sum of the weights is least.



Note that the numbers in this example can't represent distances because there's a triangle with sides 6, 7 and 14. In a geometric triangle the length of each side can't exceed the sum of the other two sides. But the numbers could refer to the capacity of pipes in a plumbing network. Or the vertices could represent a group of friends and the numbers might represent the number of Facebook messages between each pair on a given day. Although we often draw graphs pictorially don't assume that they must always represent something geometric.

There's a path in the above graph that goes through each vertex, returning to the start, whose total length is only 69. Can you find it? Is this the best possible?

Weighting in graphs can apply equally well to directed graphs. For example if some of the roads in the above example were one-way we would have a weighted directed graph. What now is the shortest path that visits each vertex and returns to the start?

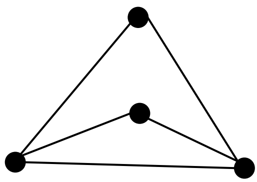


In a graph we say that two vertices are **adjacent** if there's an edge between them.

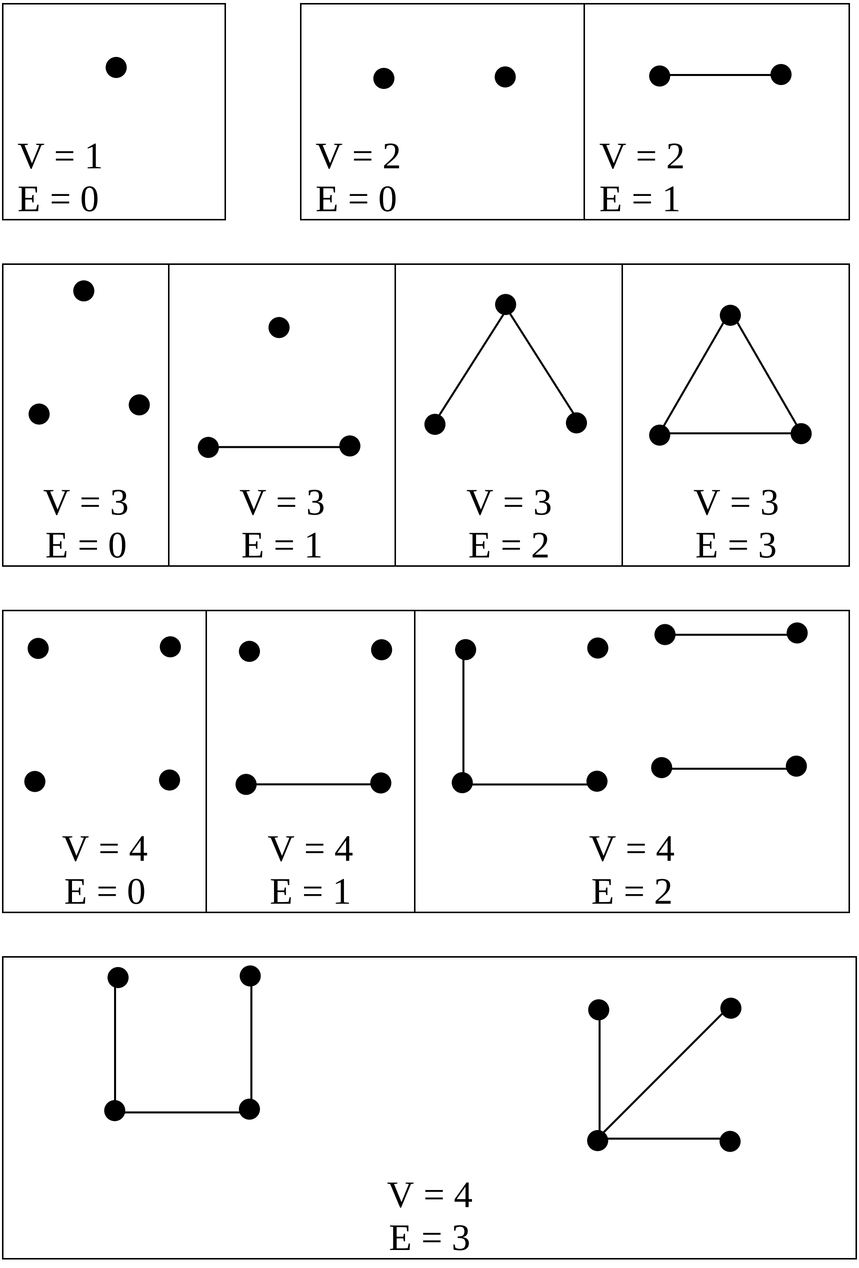
Example 7: In Example 5, vertices 1 and 2 are adjacent but 2 and 4 are not.

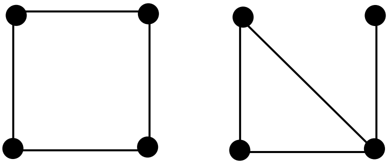
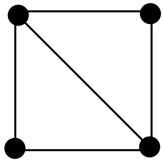
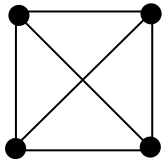
Two graphs X and Y are **equivalent** if there's a 1-1 and onto map $F: X \to Y$ such that V_1 and V_2 are adjacent in X if and only if $F(V_1)$ and $F(V_2)$ are adjacent in Y.

Example 8: The two graphs in Example 5 are equivalent. And both are equivalent to the following graph.



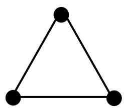
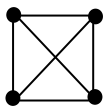
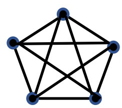
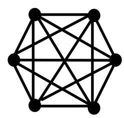
Example 9: The following list contains all the undirected graphs with 4 vertices or less. They have been systematically classified according to the number of vertices, V and edges, E.


V = 1
E = 0
V = 2
E = 0
V = 2
E = 1
V = 3
E = 0
V = 3
E = 1
V = 3
E = 2
V = 3
E = 3
V = 4
E = 0
V = 4
E = 1
V = 4
E = 2
V = 4
E = 3

		
V = 4 E = 4	V = 4 E = 5	V = 4 E = 6

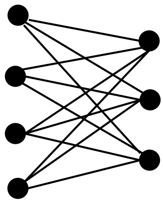
The **complete graph** on *n* vertices, denoted by $\mathbf{K}_n$, is the graph where every vertex is adjacent to every other vertex. The number of edges in a complete graph on n vertices is clearly the binomial coefficient $\binom{n}{2}$.

Example 10: The following are the complete graphs on 6 vertices or less.

$\mathbf{K}_1$	$\mathbf{K}_2$	$\mathbf{K}_3$	$\mathbf{K}_4$	$\mathbf{K}_5$	$\mathbf{K}_6$
					

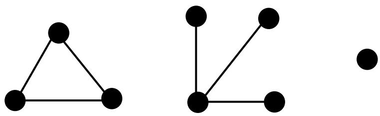
Another important family of graphs consists of the graphs $\mathbf{K}_{m,n}$ for various values of *m* and *n* (they don't have a name, just a symbol). The graph $\mathbf{K}_{m,n}$ has $m + n$ vertices divided into two subsets, one of size *m* and the other of size *n*. Every vertex in one subset is adjacent to every vertex in the other, but there are no edges connecting two vertices within the same subset.

Example 11: K_{42}:

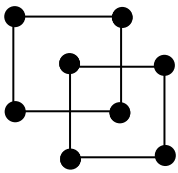


An undirected graph is **connected** if every pair of distinct vertices are connected by a path. If a graph is not connected the various parts that are connected are called its **components**. (A connected graph has just one component.

Example 12: The following graph is not connected. It has 3 components.



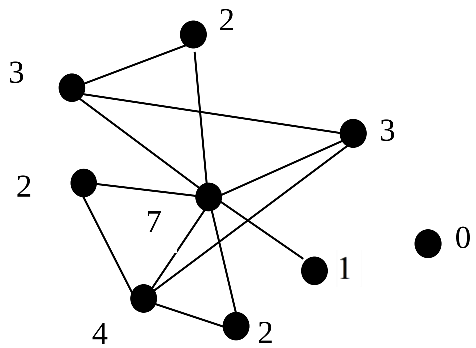
A badly drawn picture of a non-connected graph might make the graph appear connected, but note that the following graph has 2 components.



§6.3. The Degree of a Vertex

The **degree** of a vertex in an undirected graph is the number of vertices adjacent to it (in other words it is the number of edges that have that vertex as one of its endpoints). We denote the degree of vertex V by **deg(V)**.

Example 13: In the following graph the degrees of the vertices are indicated.



Notice that there are 9 vertices, 12 edges and the sum of the degrees is 24. Is there a connection between these numbers?

Theorem 1: If G is a graph with V vertices and E edges, each degree is at most V – 1 and the sum of the degrees is 2E.

Proof: The first part of this theorem is obvious. Each edge contributes 2 to the sum of the degrees, one for each end. Hence the sum of the degrees is twice the number of edges. ✋☺

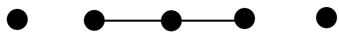
Example 14: Prove that there is no graph whose vertices have degrees 1, 2, 2, 2, 3, 4, 4, 5.
Solution: The sum of the degrees is 23, which is odd. Hence such a graph is impossible.

Example 15: Prove that there is no graph whose vertices have degrees 1, 2, 2, 2, 3, 4, 4, 8.
Solution: Since there are 8 vertices the degree of each vertex must be at most 7.

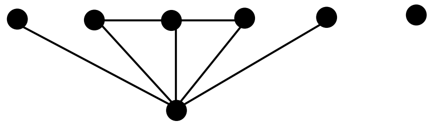
Example 16: Prove that there is no graph whose vertices have degrees 1, 1, 2, 3, 3, 3, 6, 7.
Solution: Since there are 8 vertices the vertex of degree 7 must be at most adjacent to every other vertex. Removing this vertex, and the 7 edges at that vertex we would get a subgraph with 7 vertices and degrees 0, 0, 1, 2, 2, 2, 5. Removing the vertices of degree 0 we would get a graph with degrees 1, 2, 2, 2, 5. This is impossible because there are only 4 vertices for the vertex of degree 5 to be adjacent to.

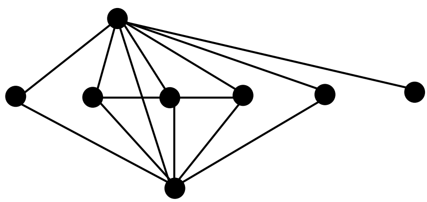
Example 17: Draw a graph G whose vertices have degrees 1, 2, 2, 3, 3, 4, 6, 7.
Solution: There are 8 vertices and so the vertex of degree 7 must be adjacent to every other vertex. Removing this vertex, and the 7 edges at that vertex we would get a subgraph H with 7 vertices and degrees 0, 1, 1, 2, 2, 3, 5. Removing the vertex of degree 0 we would get a graph with degrees 1, 1, 2, 2, 3, 5. The vertex of degree 5 would

be adjacent to every other vertex in this subgraph. Hence, as before, we reach a graph K with degrees 0, 0, 1, 1, 2. Clearly this graph must be:

K 

Working backwards we see that H and G must be as follows.

H 

G 

By examining the way the solution was obtained we can see that all graphs whose 8 vertices have degrees 1, 2, 2, 3, 3, 4, 6, 7 must be equivalent.

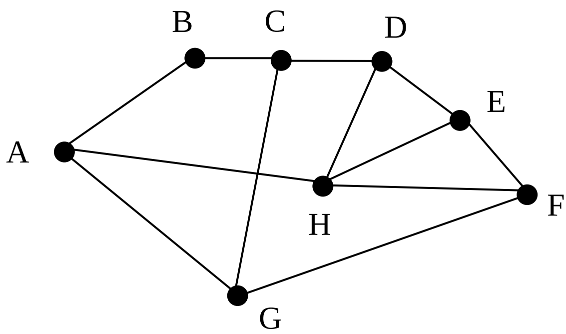
§6.4. Walks, Paths and Cycles

A **walk** in a graph (directed or undirected) is a sequence of edges

$$(V_0, V_1), (V_1, V_2), \ldots , (V_{n-1}, V_n).$$

Its **length** is the number of edges, *n*. If the edges are distinct the walk is called a **path**. If, in addition, $V_n = V_0$ the path is called a **cycle**.

Example 18: In the following graph



the sequence AB, BC, CG, FH, HD, CB is a walk. We can write it more concisely as ABCGFHDCB. Because the edge BC is repeated (even though it is in a different direction) it is not a path. Think of a walk as something you might do along a network of concrete paths in a garden while a path the actual concrete strip.

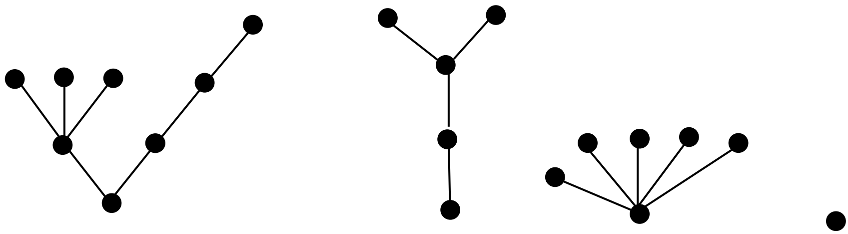
The walk ABCFH is a path since no edge is repeated. Clearly a walk can be of any length while the length of a path is at most the number of edges. The walk BCGFHAB is a cycle.

§6.5. Trees and Forests

A **forest** is an undirected graph with no cycles and a **tree** is a connected forest. The components of a forest are trees, and they can be drawn in such a way that they resemble the branches of actual trees.

In a tree there's a unique path from any vertex to any other vertex. A **leaf** of a tree is a vertex of degree 1. Often when we draw a tree one of the 'leaves' looks more like the root of the tree, but it is no different to any other leaf. Every tree, except for the tree with 1 vertex, has at least 2 leaves.

Example 19: The following is a forest with 4 components (each of which is a tree), 20 vertices and 16 edges.



There's a connection between these three numbers. For a forest the number of components plus the number of edges is equal to the number of vertices.

Theorem 2: If a forest has C components, V vertices and E edges then

$$V = E + C.$$

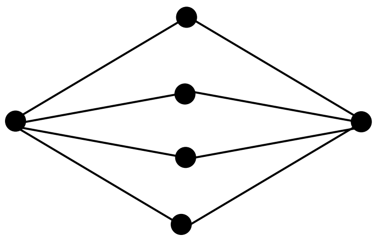
Proof: Choose an edge, and let its endpoints be A, B. Removing this edge (but not the vertices) will split one of the components into two. If it remained connected there would be a path from A to B that does not use the deleted edge, and so in the original graph there would be a cycle. But forests don't have any cycles.

The new graph will have one less edge and one more component, so E + C will remain constant. We can continue removing edges until there are none left and we have V vertices and no edges.

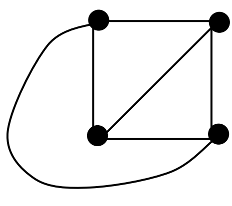
Each vertex will now be in a separate component and so we will have V components, so E + C will be equal to V. Throughout the process neither V nor E + C will have changed so V = E + C for the original graph.✋☺

§6.6. Euler's Formula

A graph is defined to be **planar** if it has only one component and can be drawn on a plane so that edges don't cross. So $\mathbf{K_{3,3}}$ is not planar. But $\mathbf{K_{4,2}}$ is:

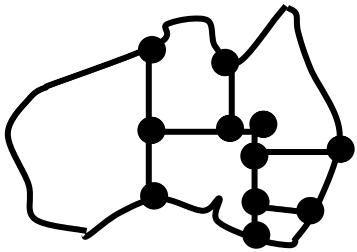


So is $\mathbf{K_4}$:

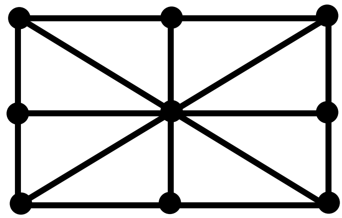


When a planar graph is drawn on a plane in such a way we call it a **map**. The regions enclosed by the edges are called **faces**. We often denote the numbers of vertices, faces and edges by V, F and E respectively.

Example 19: Some examples of maps, together with the numbers of vertices, faces and edges.



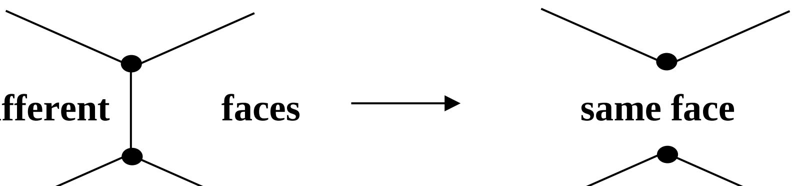
V = 11
F = 6
E = 16



V = 9
F = 8
E = 16

Note that in each case V + F – E = 1. This is always the case for planar graphs.

Theorem 3: (EULER'S FORMULA ON A PLANE)
For any connected planar map: V + F – E = 1.
Proof: Define the Euler characteristic of a map to be
$\chi = V + F - E$.
If we remove an edge from a map that separates two faces and combine those two faces the value of χ will be unchanged.

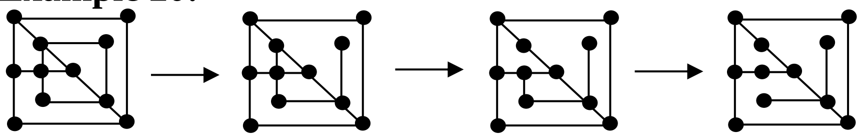


$$\left.\begin{array}{r} V \to V \\ F \to F - 1 \\ E \to E - 1 \end{array}\right\} \quad \chi \to \chi.$$

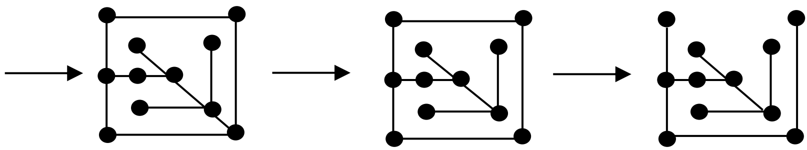
We continue this operation until there is only one face remaining. Applying the above operation we can convert any map to a tree. If there are *n* edges in this tree there are *n* + 1 vertices by Theorem 3 and so:

$$\chi = (n + 1) + 0 - n = 1.$$ ✋☺

Example 20:

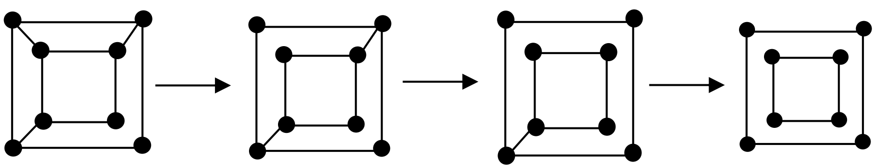


V = 11, F = 6 V = 11, F = 5 V = 11, F = 4 V = 11, F = 3



V = 11, F = 2 V = 11, F = 1 V = 11, F = 0

Example 21: Explain why the following example does not contradict Theorem 4.



Solution: For the last map:

$$V + F - E = 8 + 2 - 8 = 2.$$

Solution: In deleting edges we must ensure that they are the boundary of separate faces. This ensures that the graph remains connected. In this example the last edge that was removed does not do this.

Since a disk can be cut out of a sphere, any planar graph can be drawn on a sphere without edges crossing. On the other hand, if we can draw a graph on a sphere, without edges crossing, we can cut out a small hole in the middle of one of the faces and stretch the rest flat. In this way we will have a picture of the graph on a plane.

So a graph is planar if and only if it can be drawn, without crossings, on a sphere. This is useful because often a sphere is more convenient to work with. In this case we include the outside of the graph as a face. Indeed it isn't really possible to identify any one face as being 'outside'.

Theorem 4: (EULER'S FORMULA ON A SPHERE)
For any connected map drawn on a sphere:

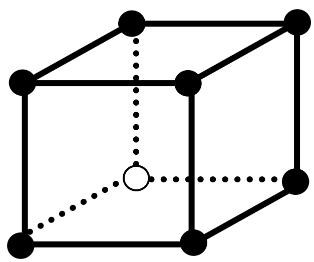
$$V + F - E = 2.$$

Proof: When a map is drawn on a sphere we have one extra face. ✋☺

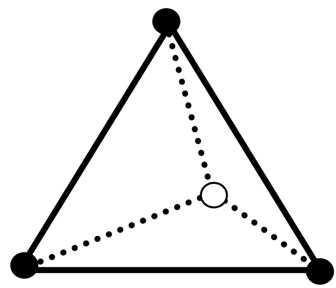
A **polyhedron** (plural 'polyhedra') is a surface in 3 dimensions made up of planar faces, each of which is a polygon. These faces meet in straight edges and the edges meet in vertices. The network of vertices and edges is a graph and if the surface is expanded to a surface (imagine that the faces are made of some material that stretches and it is pumped up) we get a planar graph drawn on a sphere. So for polyhedral:

$$V + F - E = 2.$$

Example 22: Maps on a sphere:



V = 8
F = 6
E = 12
$\therefore \chi = 8 + 6 - 12 = 2.$



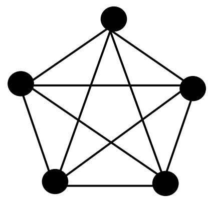
V = 4
F = 4
E = 6
$\therefore \chi = 5 + 5 - 8 = 2.$

§6.7. Planar Graphs

To prove that a graph is planar we can simply draw it, with edges not crossing. But how do we show that a graph, such as, $\mathbf{K_{3,3}}$ or $\mathbf{K_5}$ is *not* planar? The technique discussed here is to work out the average number of edges per face and compare this to the smallest number of edges per face. But wait a minute. How can we count the faces unless we can draw it without edges crossing?

That's true, but the answer is to use Euler's formula for a sphere: V + F – E = 2

Example 23: $\mathbf{K_5}$ is not planar.



Proof: For $\mathbf{K_5}$ we have V = 5 and E = 10.

Suppose that $\mathbf{K_5}$ is planar. Then embedding it in a sphere we can deduce that the number of faces must be:

$$F = 2 + E - V = 7.$$

The average number of edges per face must therefore be $\frac{2E}{F} = \frac{20}{7} = 2\frac{6}{7}$.

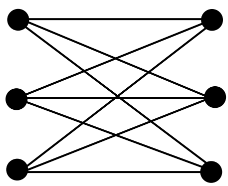
Why $\frac{2E}{F}$ and not just $\frac{E}{F}$? The reason is that every edge is associated with two faces – one on each side. So if you were to split each edge lengthwise, so that each half edge was associated with only one face, you'd have 2E half edges to share among the F faces.

Now we wanted to prove that $\mathbf{K_5}$ can't be embedded in a sphere and we started out by supposing that it can be. We're clearly looking for a contradiction. So what's contradictory about the average number of edges per face being $2\frac{6}{7}$?

What is wrong is that it's less than 3. Every face must be surrounded by at least 3 edges. (A face bounded by 2 edges would require that the two edges connect the same two vertices, and a face bounded by just 1 edge would mean that the graph has a loop.)

Now the average of a collection of numbers can't be less than the smallest of them. So here we have our contradiction!

Example 24: $\mathbf{K_{3,3}}$ is not planar.



Proof: Here V = 6 and E = 9.
Suppose that $\mathbf{K_{3,3}}$ can be embedded in a sphere. The resulting map would have to have F faces where:

$$6 + F - 9 = 2,$$

that is it must have 5 faces.

The average number of edges per face would therefore be $\frac{18}{5} = 3\frac{3}{5}$.

That's not less than 3, so where's the contradiction? The contradiction is that it's less than 4. You see, in this graph there are no cycles of length 3. Each edge takes you from one set of vertices to the other. Going along another edge must take you back to a different vertex in the first set. The smallest cycles in this graph therefore have length 4. The boundary of a face must be a cycle in the graph. So the smallest number of edges for each face is 4. But the average of these numbers is less than 4. This can't be, and so we have our contradiction.

The **girth** of a graph is the length of the shortest cycle. The girth of $\mathbf{K_5}$ is 3 but the girth of $\mathbf{K_{3,3}}$ is 4. We get a contradiction if the average number of edges per face is less than the girth.

Theorem 5: If a connected graph has V vertices, E edges and a girth of g such that:

$$\frac{2E}{2 + E - V} < g$$

then the graph is not planar.