DISCRETE MATHEMATICS

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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

INTRODUCTION

These notes are devoted to several topics which, while firmly within mathematics, are of great interest to computing scientists.

They begin with an account of logic. This underpins mathematics, as well as being important in several branches of both the hardware and software sides of computing science.

Then we develop some basic number theory, the theory of divisibility and prime numbers. Once thought to be the purest (and most useless) branches of mathematics, it has become an important tool in coding theory – necessary for the security of electronic data transmission.

In chapter 3 we discuss the fundamental concepts of mathematics, such as sets, relations, functions and strings.

At this stage we discuss proofs. Proofs are a stumbling block to many students. We will investigate the nature of proof from the syntactic point of view, that is, thinking of proofs as sequences of strings of symbols related to each other in a mechanical way. This is not the whole picture since significant proofs require insight and imagination. But routine proofs — the sort that *you* might be called upon to construct — can be generated in a routine, mechanical way from the definitions and the logical structure of the statements to be proved. As well as helping you to write sound mathematical proofs this training will give you some insight into automated proofs of program correctness.

Counting is the most basic skill in mathematics, but is relevant to computing science, especially when we want to estimate the number of steps and amount of memory that certain algorithms require.

A lot of data that computer programs must process come in the form of weighted graphs. But graphs are also a fundamental concept in mathematics and have particular relevance to topology. As well as discussing the elementary theory of graphs we present several of the well-known algorithms for graphs.

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