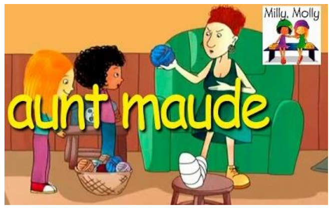
3. SETS, FUNCTIONS & RELATIONS

If I see the moon, then the moon sees me
'Cos seeing's symmetric as you can see.
If I tell Aunt Maude and Maude tells the nation
Then I've told the nation 'cos the gossiping relation
Is transitive. And even if you're on the shelf
You're loved by someone, that's yourself,
'Cos love's reflexive. And if all three,
Reflex- and Trans- and Symmetry,
I'm sure you see, it makes good sense
To call it a relation of equivalence.



Ah, when I was young it came to pass
That I learnt all this in equivalence
class.

§3.1. Sets in Mathematics and Computing Science

As we studied elementary mathematics we came to believe that the fundamental object in mathematics is the number. Then, as we began calculus, we learnt of a new type of mathematical object – the function. At first we thought of a function as a formula, or an algebraic expression, such as the function x^2. But then they tried to tell us that functions are more than just formulae. They are rules which associate with every element in one set a unique element of another. Every function has to go from one set to another.

The deeper we get into mathematics the more we become involved with sets. We learn of vector spaces and groups – these are sets on which there is some algebraic structure. In geometry we no longer deal with shapes, but rather, sets of points. And in calculus we're increasingly concerned with domains of functions and regions of convergence and sets of solutions to differential equations. The set is paramount in mathematics.

In computing science sets are one of many important data structures. In databases they are the fundamental data type.

§3.2. Defining a Set

A **set** is a collection of objects called **elements**. We write $x \in S$ if x is an element of the set S and $x \notin S$ if it is not.

Generally elements are denoted by lower case letters and sets by capitals. However since there are sets whose elements are sets, and sets of sets of sets etc., it is not always possible to maintain this distinction.

There are three main ways of describing or defining a set:

(1) Use a standard name

$\mathbb{N}$ = the set of natural numbers 0, 1, 2, ...

NOTE: Some books exclude zero from this set and consider the natural numbers to be synonymous with the positive integers.

Example 1:

$\mathbb{Z}$ = the set of integers 0, ±1, ±2, ... (from the German 'Zahlen', meaning "numbers");
$\mathbb{Q}$ = the set of rational numbers ($\mathbb{Q}$ stands for 'quotient');
$\mathbb{R}$ = the set of real numbers;
$\mathbb{C}$ = the set of complex numbers.

(2) List the elements

$\{x_1, x_2, \ldots, x_n\}$ denotes the set with the elements: $x_1, x_2, \ldots, x_n$.
Strictly speaking this notation can only be used for finite sets but if there's an obvious pattern we can indicate certain infinite sets this way.

Example 2:

{4, 5, 13} is the set whose elements are the integers 4, 5 and 13;
{0, 1, 4, 9, ... } indicates the set of square integers.

(3) Describe a property that characterises the elements.

$\{x \in S \mid Px\}$ denotes the set of all elements x, in S, for which the statement Px is true. If the set S is understood we often omit it and write $\{x \mid Px\}$.

Example 3:

$\{x \in \mathbb{Z} \mid x > 0\}$ is the set of positive integers, commonly denoted by $\mathbb{Z}^+$;

$\{x \in \mathbb{Z} \mid 3 < x < 7\} = \{4, 5, 6\}$;

$\{x \in \mathbb{R} \mid x^2 < 1\}$ denotes the open interval $(-1, 1)$ i.e. all real numbers x such that $-1 < x < 1$;

$\{n \mid \exists q[n = 7q]\}$ denotes the set of all multiples of 7.

NOTES:

(1) The symbol x in the notation $\{x \mid \mathrm{Px}\}$ is a 'dummy' variable. It can be replaced throughout by any other symbol not otherwise used. Thus $\{x \mid \mathrm{P}x\} = \{r \mid \mathrm{P}r\}$.

(2) $(y \in \{x \mid \mathrm{P}x\}) \leftrightarrow \mathrm{P}y$, that is, y belongs to a set if and only if it has its defining property.

(3) When we describe a finite set by listing the elements the order does not matter. Also any repetitions are ignored. For example: $\{3, 1, 2, 4\} = \{1, 2, 3, 4\} = \{1, 2, 2, 3, 4, 4, 4\}$.

(4) It was once naively thought that for every property there must be a set, but this can lead to certain paradoxes. The most famous is the Russell Paradox named after the philosopher and mathematician Bertrand Russell. If $\mathrm{S} = \{x \mid x \notin x\}$ then $\mathrm{S} \in \mathrm{S}$ if and only $\mathrm{S} \notin \mathrm{S}$. [If $\mathrm{S} \in \mathrm{S}$ it satisfies the defining property and so $\mathrm{S} \notin \mathrm{S}$. If $\mathrm{S} \notin \mathrm{S}$ it does not satisfy the defining property and so $\mathrm{S} \in \mathrm{S}$!] We clearly cannot allow a mathematical system to exist in

which such paradoxes are possible. Some time ago, some mathematicians interested in the foundations of mathematics established axioms for set theory in which there are precise restrictions on which properties are allowed to give rise to a set. Fortunately such deep problems are far removed from the coal-face of useful mathematics and we can safely ignore them.

Two sets are defined to be **equal** if every element of one set is an element of the other. In symbols:

$$S = T \leftrightarrow \forall x[x \in S \leftrightarrow x \in T]$$

Example 4: $\{x \in \mathbb{Z} \mid x^2 < 2\} = \{-1, 0, 1\}$
$= \{x \in \mathbb{R} \mid x^3 = x\}$

A set S is a **subset** of a set T if every element of S is an element of T. We write **S ⊆ T** to denote the fact that S is a subset of T. In symbols this is very similar to the definition of equality of sets:

$$S \subseteq T \leftrightarrow \forall x\,[x \in S \rightarrow x \in T]$$

In particular every set is a subset of itself. Subsets *S* which are not the whole set *T* are called **proper** subsets. We denote this by **S ⊂ T**.

A very important set is the **empty** set. This is the set with no elements and can be described by an empty list { } or an impossible property, such as $\{x \mid x \neq x\}$. Note that there is only one empty set. For example, the empty set of integers whose square is 2 is the same as the empty set of triangles with four sides. The statement “mermaids don’t exist” can be expressed by saying that “the set of mermaids is the empty set”. The empty set has its own special symbol, $\varnothing$.

Like the number 0, it’s an extremely useful concept. If we want to prove that a given property *P* can never hold we can consider the set $S = \{x \mid Px\}$ and then prove that $S = \varnothing$.

Another important set is the ‘universe’ a set consisting of all the elements with which we are concerned in a given context. For example if we are considering sets of integers we would take our universe to be the set of *all* integers.

§3.3. Basic Set Functions

We now list a large number of definitions of functions which assign to each set *S*, or pair of sets *S* and *T*, a certain set.

Intersection:

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

Union:

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

Difference:

$$S - T = \{x \mid x \in S \text{ and } x \notin T\}$$

Complement:

$$-S = \{x \mid x \notin S\} = U - S$$

where U is the universe in which we are working.

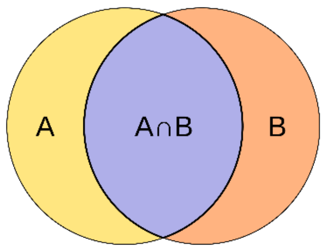
Example 5

Suppose S = {1, 2, 3} and T = {1, 3, 5, 7}.

Then $S \cap T = \{1, 3\}$; $S \cup T = \{1, 2, 3, 5, 7\}$; $S - T = \{2\}$; $T - S = \{5, 7\}$

If our universe is the set of all integers and S is the set of even numbers, –S is the set of all odd numbers.

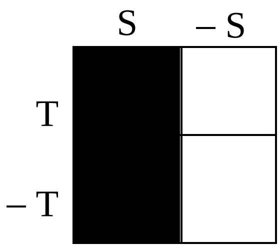
§3.4. Venn Diagrams



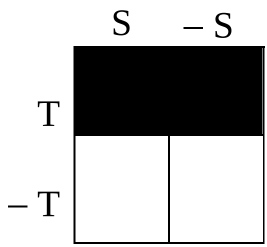
Some of these concepts can be illustrated by diagrams where sets are represented by regions drawn in the plane and elements by points inside them.

Traditionally Venn diagrams represent sets by overlapping ovals. The intersection, for example, is represented by the overlapped region.

However, the simplest system is to divide a rectangle into two rows and two columns. The left side represents one set and the right side, its complement. Then the top half can represent a second set and the bottom, its complement.



S



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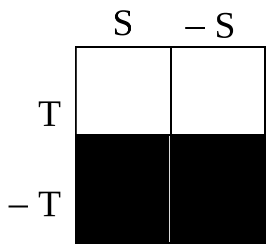
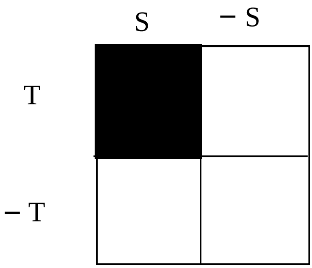
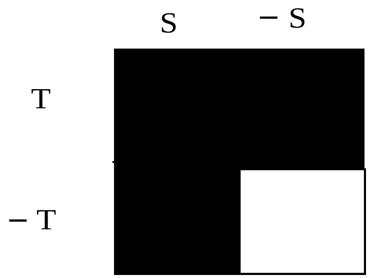


–T

Example 6:

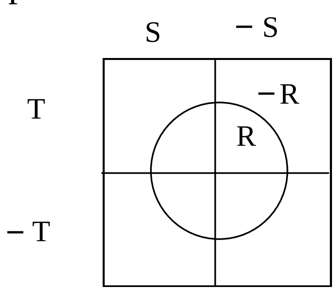


INTERSECTION



UNION

If there is a third set we can simply draw a circle in the middle, cutting across all four regions. The region inside the circle represents the third set and the outside represents its complement.



§3.5. Extended Set Functions

Cartesian Product:

$$S \times T = \{(x, y) \mid x \in S \text{ and } y \in T\}$$

= the set of all ordered pairs whose components come from S and T respectively

$$S \times T \times R = \{(x, y, z) \mid x \in S, y \in T, z \in R\} \text{ etc.}$$

Cartesian Power:

$$S^n = S \times S \times \ldots \times S \quad (n \text{ factors})$$

Power Set:

$$\wp(S) = \{A \mid A \subseteq S\} = \text{the set of all subsets of } S$$

Example 7:

Suppose $S = \{1, 2, 3\}$ and $T = \{1, 3, 5, 7\}$. Then

$S \times T = \{(1, 1), (1, 3), (1, 5), (1, 7), (2, 1), (2, 3), (2, 5), (2, 7), (3, 1), (3, 3), (3, 5), (3, 7)\}$

$S \cap T = \{1, 3\}$

$(S \cap T)^3 = \{(1, 1, 1), (1, 1, 3), (1, 3, 1), (1, 3, 3), (3, 1, 1), (3, 1, 3), (3, 3, 1), (3, 3, 3)\}$

$\wp(S) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$

NOTES:

(1) While the order of the elements of a set does not matter there is often a particular order that is more natural than the others. The advantage of using a systematic order is

that it ensures that every combination has been accounted for. Look carefully at the order in which we have listed the elements in the above example.

(2) Venn Diagrams cannot be used to depict these extended set functions.

§3.6. Relations

"Jack is my mother's uncle." That is what we usually think of when we talk about relations – people with whom we are related. But it is the relationship itself that mathematicians would call a *relation*.

A **relation** is a statement involving two variables, or two gaps, where the names of people or things can be inserted.

So '*x* is the mother of *y*', or just 'mother of' is one such relation.

In mathematics there are numerous relations such as 'is less than, 'is parallel to', 'is a power of .

We can denote the fact that *x* has the relation with *y* by *x*R*y*. This reflects the word order in natural language, and indeed where we have invented symbols for particular relations in mathematics we usually use this format, such as “$x < y$”.

But a relation, as well as being a connection between two sets, can also be considered as a set in itself. Suppose R is a relation such that $x\text{R}y$ makes sense for $x \in$ S and $y \in$ T. We can represent R by the set of ordered pairs (x, y) for which the relation holds, that is
$\{(x, y) \mid x\text{R}y\}$.

Now S × T denotes the set of all ordered pairs of the form (x, y) with $x \in$ S, $y \in$ T. So the set of ordered pairs for which $x\text{R}y$ (is true) is just a subset of S × T. This gives us an alternative definition of a relation.

A **relation from a set S to a set T** is any subset of S × T.

A **relation on a set S** is a relation from S to S i.e. a subset of S × S.

Example 8: If S = {1, 2, 3, 4} then the relation normally written “$x < y$” would be:

$$\{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)\}$$

and the relation “$x = y^n$ for some integer $n \geq 1$” becomes: $\{(2, 2), (3, 3), (4, 2), (4, 4)\}$.

When a computer makes reference to a relation such as “<” it certainly doesn’t consult an array of ordered pairs. But when dealing with the relation “student x is enrolled in course *y*" we might store this information in a database as an array of ordered pairs.

Example 9:

STUDENT	COURSE
AARON Adam	MATHS 1
AARON Adam	PHYSICS 1
ANTON Tom	MATHS 1

In mathematical notation we would write this relation as

{(AARON Adam, MATHS 1),
(AARON Adam, PHYSICS 1),
(ANTON Tom, MATHS 1)}

§3.7. The Sum and Product of Relations

If R and S are relations on the set X then the **sum** of R and S is the relation R + S defined on X by: $x(R+S)y$ if xRy **or** xSy. As sets, this is simply the union: $S + T = S \cup T$.

Example 10: The relation 'spouse of' means 'husband or wife of'. If H = 'husband of' and W = 'wife of' then H + W is the relation 'spouse of'.

Example 11: If L is the relation '<' on the set of real numbers, and E is the relation of equality, then L + E is the relation '≤'. (We could write this as '< + = = ≤' but this would be a little confusing!)

If R and S are relations on the set X then the **product** of R and S is the relation RS defined on X by: $x\mathrm{RS}y$ if there is some $u \in \mathrm{X}$ such that $x\mathrm{R}u$ and $u\mathrm{S}y$.

In symbols, $\mathrm{RS} = \{(x, y) \mid \exists u[x\mathrm{R}u \wedge u\mathrm{R}y]\}$.

Example 12: Jack is married to Mary and Mary's mother, Ruby, is therefore Jack's mother-in-law. The relation 'mother-in-law of ' is built up from the simpler relations of 'spouse of ' and 'mother of '. Ruby is the mother-in-law of Jack because there exists a person, Mary, such that Ruby is the mother of Mary and Mary is the spouse of Jack. There's a chain between Ruby and Jack and Mary is the link.

If H + W denotes 'spouse of ' and M = 'mother of ' then M(H+ W) denotes the relation 'mother-in-law of '.

If R is a relation on a set X, $\mathrm{R}^2 = \mathrm{RR}$, $\mathrm{R}^3 = \mathrm{RRR}$ etc.

Example 13: Mary's parents are Ruby and Ted. Ted is the husband of Mary's mother. So with H and M defined above, HM = 'father of '. And M + HM = 'parent of '.
So $(\mathrm{M} + \mathrm{HM})^2$ = 'grandparent of '.

§3.8. Equivalence Relations

Like all mathematical objects, relations can be classified according to whether they satisfy certain properties. The three most important properties for a relation R on a set S are the **reflexive**, **symmetric** and **transitive** properties.

R is **reflexive** if xRx for all x.
R is **symmetric** if xR$y \to y$Rx for all x, y.
R is **transitive** if xRy and yRz $\to x$Rz for all x, y, z.

Example 14: $x < y$ on the set of integers.
Not Reflexive: For example $1 < 1$ is FALSE.
Not Symmetric: For example $1 < 2$ is TRUE but $2 < 1$ is FALSE.
Transitive: If $x < y$ and $y <$ z it follows that $x <$ z.

Example 15: Suppose that xRy is defined to mean that $x = y^n$ for some $n \in \mathbb{N}$ is a relation on the set $\mathbb{N}$.
Reflexive: Since $x = x^1$ for all x.
Not symmetric: Now 4R2 (as $4 = 2^2$) but it is not true that 2R4 ($2 \neq 4^n$ for any positive n).
Transitive: Suppose xRy and yRz.
Then $x = y^n$ for some $n \in \mathbb{N}$ and $y = z^m$ for some $m \in \mathbb{N}$.
(We mustn't fall into the trap of assuming that the power is the same in each case.)
Hence $x = (z^m)^n = z^{mn}$. Since $mn \in \mathbb{N}$ it follows that xRz.

Example 16: Suppose xRy is defined to mean that

$$|x - y| < 3$$

Reflexive:
Since for all x, $|x - x| = 0$ which is less than 3.
Symmetric: Since $|y - x| = |x - y|$.

Not Transitive: For example 1R3 and 3R5 but it is not true that 1R5.

Example 17: Suppose $x\mathrm{R}y$ is defined to mean that $x - y$ is even, defined on the set $\mathbb{Z}$.
Reflexive : For all x, $x - x = 0$ which is even.
Symmetric: Suppose $x\mathrm{R}y$.
This means that $x - y$ is even, say $x - y = 2h$ for some $h \in \mathbb{Z}$.
Then $y - x = -2h = 2(-h)$ which is even, so $y\mathrm{R}x$.
Transitive: Suppose $x\mathrm{R}y$ and $y\mathrm{R}\mathrm{z}$.
This means that $x - y = 2h$ for some $h \in \mathbb{Z}$ and $y - \mathrm{z} = 2k$ for some $k \in \mathbb{Z}$.
Now $x - \mathrm{z} = (x - y) + (y - \mathrm{z}) = 2h + 2k = 2(h + k)$ which is even.
Hence $x\mathrm{R}\mathrm{z}$.

An **equivalence relation** is a relation on a set that is reflexive, symmetric and transitive. Example 17 is an example of an equivalence relation.

§3.9. Equivalence Classes

If R is an equivalence relation and $x \in \mathrm{S}$, we define

$$\mathbf{[}\boldsymbol{x}\mathbf{]_R} = \{y \in \mathrm{S} \mid x\mathrm{R}y\},$$

that is, the set of all elements of S that are equivalent to x under the relation R.

We call $[x]_R$ the **equivalence class** containing *x*. Often we omit the subscript R.

NOTE: The fact that $[x]_R$ does indeed contain *x* follows from the reflexive property.

To find the equivalence classes for a given equivalence relation use the following simple algorithm. Choose an element *x* which has not yet been included in an equivalence class (any element will do to begin with). List all the elements that are related to *x*. These form an equivalence class. Now choose another element not yet listed. Continue until all elements have been included.

Equivalence classes are somewhat like families. Those in the same class are related – those in different classes are not. Human families don't provide a perfect analogy with equivalence classes since being related is not an equivalence relation. (The relation of being related is not transitive since my cousin has cousins who I would not consider as being related to me.)

Example 18: Consider the set of railway stations of the world with xRy meaning that you can travel by rail from station *x* to station *y* (possibly having to change trains one or more times). The equivalence classes are the connected railway networks. One would be the government railways of mainland Australia. Another would have been the Tasmanian Railways, but that is now the empty set. Some privately owned railways would form little equivalence classes. Until recently the British Rail network in England, Scotland and Wales formed an entire equivalence class on its own, but with the opening of the Channel Tunnel it now forms part of a larger network that extends across Europe.

Example 19: Let S = {1,2,3} and let
R = {(1,1), (1,3), (2,2), (3,1), (3,3)}.
One can show that R is an equivalence relation. The equivalence classes are:

$$[1] = \{1,3\} = [3] \text{ and } [2] = \{2\}.$$

Theorem 1: Let R be an equivalence relation.
Then $x \in [y] \leftrightarrow [x] = [y]$
Proof: Suppose that $[x] = [y]$. Since $x \in [x]$, $x \in [y]$.
Conversely, suppose that $x \in [y]$. Then yRx (and by the symmetric property, xRy). We must show that $[x] = [y]$.
We first show that $[x] \subseteq [y]$.
Suppose that $z \in [x]$. Then xRz.
Since yRx, it follows from the transitive property that yRz and hence zRy by the symmetric property.
Thus $z \in [y]$, and so $[x] \subseteq [y]$.
Similarly, $[y] \subseteq [x]$, and so $[x] = [y]$.
Theorem 2: For all $x, y \in$ S, either $[x] \cap [y] = \varnothing$ or $[x] = [y]$.
Proof: Suppose that $[x]$, $[y]$ are not disjoint.
Let $z \in [x] \cap [y]$.
Then $z \in [x]$ and $z \in [y]$. By the previous theorem,
$[x] = [z]$ and $[z] = [y]$ and so $[x] = [y]$.

We have proved that any two distinct equivalence classes are disjoint. So:

> IF R IS AN EQUIVALENCE RELATION ON A SET S THEN *S* IS THE DISJOINT UNION OF ITS EQUIVALENCE CLASSES

This means that S can be chopped up into non-overlapping equivalence classes so that elements are equivalent if and only if they belong to the same class.

§3.10. Functions

A **function** f from a set S to a set T is the pair of sets (S, T) together with a rule that associates with each element $x \in S$ a unique element of T, written $f(x)$. This element is called the **image** of x under f.

We indicate that f is a function from S to T by writing f:S→T. The set S is called the **domain** of f and T is called the **codomain**. Other words that are used instead of 'function' are 'map', 'operator' and 'transformation'.

Example 20: f: S → S where S is the set of people who are now living or who have died, with $f(x)$ defined to be "the father of x". But note that "the son of x" does not define a function on S because, although everyone has a unique (genetic) father, not everyone has a unique son and many people in S have no sons at all.

The simplest way to describe a function from one finite set to another is by means of a table of values.

Example 21: f:{1, 2, 3, 4}→{1, 2, 3, 4} where f is defined by means of the following table of values:

x	$f(x)$
1	2
2	1
3	3
4	2

You don't have to discover any pattern in this table, or find a formula, before you can call f a function. The rule here is simply to look up the table. The second column could have been filled up in any way, as long as no symbol other than 1, 2, 3, 4 is used.

Example 22: There are 8 functions from {1, 2, 3} to {A, B} viz.

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$	$f_7(x)$	$f_8(x)$
1	A	A	A	A	B	B	B	B
2	A	A	B	B	A	A	B	B
3	A	B	A	B	A	B	A	B

Notice the systematic way in which these functions have been listed.

§3.11. One-to-one and Onto Functions

A function f:S $\rightarrow$T is **1-1** (pronounced "**one to one**") if

$$f(x) = f(y) \rightarrow x = y.$$

The **image** of a function f:S$\rightarrow$T is $\{f(x) \mid x \in S\}$.
It is denoted by **im f**.

A function f:S$\rightarrow$T is **onto** if $t \in T \rightarrow \exists s[f(s) = t$ and $s \in S]$.

A **permutation** on a set S is a 1-1 and onto function f from a set to itself.

Example 23: Consider the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = x + 1;$$
$$g(x) = x^2 + 1;$$
$$h(x) = x^3 + x^2;$$
$$k(x) = e^x.$$

Then f is 1-1 and onto, g is neither h is onto but not 1-1 and k is 1-1 but not onto.

Example 24: The function f(x) = 'the father of x' is not 1-1 since there are different people with the same father. In the year 1900 the function h(x) = 'the current husband of x' was a function from the set of married women to the set of married men. (These days, with gay marriage, things are more complicated.) In the state of Utah, where polygamy was practised, the function was not 1-1, but in Australia it was.

If P = {all people living today}, M = {all men who have ever lived} then f:P→M is not onto because, for example, the image of f doesn't include the Pope (we believe that he has never fathered a child). If F = im f then c: But c:F→P defined to be 'eldest child of' is onto.

Example 25: The six permutations on the set {1,2,3} are given by the following table:

x	$f_1(x)$	$f_2(x)$	$f_3(x)$	$f_4(x)$	$f_5(x)$	$f_6(x)$
1	1	1	2	2	3	3
2	2	3	1	3	1	2
3	3	2	3	1	2	1

Note the systematic order in which these permutations have been listed.

§3.12. Counting Finite Sets

The **cardinal number** of a finite set is its number of elements. We denote the cardinal number of S by |S|). For example |∅| = 0.

Example 26: If S = {1, 2, 3} and T = {3, 5} then:
|S| = 3, |T| = 2, |S ∩ T| = 1, |S ∪ T| = 4, |S − T| = 2, |S × T| = 6, |℘(S)| = 8.

Theorem 3: Let S, T be finite sets. Then:

(1) $|\emptyset| = 0$;

(2) $|S \cup T| = |S| + |T| - |S \cap T|$;

(3) If S, T are disjoint then $|S \cup T| = |S| + |T|$.

(4) $|S \times T| = |S|.|T|$;

(5) $|S - T| = |S| - |S \cap T|$;

(6) $|\wp(S)| = 2^{|S|}$;

Proof: Left as an exercise.

It is possible to define the cardinal number of an infinite set, but to do this we have to invent a system of infinite cardinal numbers, each infinite, but with ever increasing size. This may surprise you, having just used the symbol ∞ for anything infinite. But there really *are* different levels of infinitude. And, moreover, it turns out that for the familiar sets we've grown up with,

$$|\mathbb{Z}| = |\mathbb{Q}| < |\mathbb{R}| = |\mathbb{C}| < |\{f: \mathbb{R} \rightarrow \mathbb{R}\}|.$$

You have already met infinite sets with three different sizes! When it comes to infinite sets we define the sums and products of infinite numbers so that (3) and (4) hold, rather than prove them. We'll discuss this further in §3.13.

Theorem 4: Let S be a finite set of size n. Then:

(1) The number of relations on S is 2^{n^2}.

(2) The number of reflexive relations on S is $2^{n(n-1)}$.

(3) The number of symmetric relations on S is $2^{\frac{1}{2}n(n+1)}$.

Proof:

(1) A relation on S is a set of ordered pairs. There are thus two choices for each ordered pair – the relation holds for that pair or it doesn't. Since there are n^2 ordered pairs there are 2^{n^2} choices, that is 2^{n^2} relations altogether.

(2) We can represent these ordered pairs by putting dots in an $n \times n$ table where each dot can be changed to a tick or a cross depending on whether or not the relation holds in that case. For a reflexive relation the diagonal of this table will have to consist of all ×'s leaving $n^2 - n$ dots for which there is a choice of a tick or a cross. Thus the number of reflexive relations is $2^{n(n-1)}$.

(3) For a symmetric relation the table of ticks and crosses has to be symmetric about the diagonal. So we are free to choose a tick or a cross for the dots above the diagonal and also those on the diagonal. But having done so there is no further choice available below the diagonal. This gives $\frac{1}{2}n(n-1)$ positions above the diagonal and n positions on the diagonal, giving a total of $\frac{1}{2}n(n+1)$ positions altogether and so $2^{\frac{1}{2}n(n+1)}$ symmetric relations.

Example 27: The relation {(1, 1), (1, 3), (2, 2), (2, 4), (3, 1), (3, 3), (4, 2), (4, 4)} on the set
{1, 2, 3, 4} gives rise to the table:

	1	**2**	**3**	**4**
1	√	×	√	×
2	×	√	×	√

3	√	×	√	×
4	×	√	×	√

from which we can see that the relation is both reflexive and symmetric.

The transitive relations are rather difficult to count. But we can count equivalence relations by a completely different method. Every equivalence relation on *S* corresponds to a partition of *S* into equivalence classes and every partition of *S* corresponds to an equivalence relation. So we simply need to count the partitions.

Example 28:
The equivalence relation {(1, 1), (1, 4), (2, 2), (3, 3), (4, 1), (4, 4)} corresponds to the partition

{{1, 4}, (2}, {3}}.

Example 29: How many equivalence relations are there on the set {1, 2, 3, 4}?

Solution: We begin by enumerating the types of partition. The above partition consists of a pair and two singles. We can represent this type by (××)(×)(×). With 4 elements the possible types of partition are: (××××), (×××)(×), (××)(××), (××)(×)(×) and (×)(×)(×)(×).

Now we have to count the numbers of partitions of each type. For (××××) there is only one partition – all in together. For (×××)(×) there are 4 choices for the singleton. Having chosen which one goes by itself there is no further choice – all the other three go in together. For the (××)(×)(×) type we have ${}^4C_2 = 6$ ways of choosing the two singletons and so 6 partitions of that type. And there is only one partition of the type (×)(×)(×)(×).

Type	**No.**
(××××)	1
(×××)(×)	4
(××)(××)	3
(×××)(×)(×)	6
(×)(×)(×)(×)	1
TOTAL	**15**

The partitions of the type (××)(××) are a little more complicated to count. To start with we have ${}^4C_2 = 6$ choices for the first pair, with no further choice for the second. But because the two pairs have the same size they're interchangeable and so each partition would have been counted twice.

For example {1, 3}, {2, 4}} and{{2, 4}, {1, 3}} are the same partition. So we have just 3 partitions of this type.

Thus there are 15 equivalence relations on this set.

Theorem 5: Let S, T be finite sets of sizes m, n respectively.

(1) The number of functions f:S→T is n^m.

(2) The number of 1-1 functions f:S→T is $\dfrac{n!}{(n-m)!}$.

(3) If $m > n$ the number of 1-1 functions f:S→T is 0.
(4) If $m < n$ the number of onto functions f:S →T is 0.
(5) If $m = n$ the number 1-1 and onto functions f:S→T is n!.
(6) The number of onto functions f:S→T is n! times the number of equivalence relations with n equivalence classes.

Proof:

(1) For each $x \in$ S there are n choices for $f(x)$ and so n^m choices of function.
(2) If the function is required to be 1-1 the choices reduce as we move from one element to the next and so there are $n(n-1)(n-2)\ \ldots$ choices of 1-1 function. There are m factors here.
(3), (4) Note that if $m > n$ one of these factors is zero. There can be no 1-1 functions from a larger set to a smaller one just as there can be no onto function from a smaller set to a larger.
(5) In the case where $m = n$ note that a function is 1-1 if and only if it is onto and so there are n! such functions in this case (none if $m \neq n$).
(6) Finally if we have an onto function f: S → T the equivalence relation $f(x) = f(y)$ has exactly n equivalence classes. Now for each partition of S into n equivalence classes there are n! different onto functions according to the n! ways in which we can assign the n elements of T to the n equivalence classes.

Example 30: Find the number of onto functions from {1, 2, 3, 4, 5, 6} to {a, b, c}.
Solution: The types of partitions with 3 classes, and the numbers of each type, are as follows:

Type	**Number**
(××××)(×)(×)	15
(×××)(××)(×)	60
(××)(××)(××)	15 (= $^6C_2.^4C_2/3!$)
TOTAL	**90**

There are 90 partitions of 6 elements into 3 classes and 3! onto functions for each of these partitions, giving 540 onto functions in all.

§3.13. Counting Infinite Sets

It is possible to extend the notion of counting from finite sets to infinite ones, producing an arithmetic of infinite numbers. At first sight it might seem that this arithmetic is not very interesting, with just one infinite number called ∞. But in fact there are different levels of infinitude!

Counting involves two stages. We must first have a notion of 'same size' and then a collection of 'standard sets', one for each size. This is analogous to the old method of weighing objects with a set of scales. Two objects have the same weight if the scales balance with one object in each pan. We weigh objects in one by having a set of standard weights in the other.

In the case of 'same size' we use the existence of a 1-1 and onto function from one set to the other.

Sets *S*, T have the same size if there exists a 1-1 and onto function *f*:S→T.

This is how we first learnt to count in kindergarten. In pointing to each object in turn we were setting up such a 1-1 and onto function from the set to be counted to one of the standard sets $\{1, 2, \ldots, n\}$. These standard sets conveniently nest, one inside the other and any set in 1-1 correspondence with $\{1, 2, \ldots, n\}$ is said to have size *n*.

The same definition of 'same size' applies to infinite sets too, but the surprising fact is that not all infinite sets have the same size. So we shall have to abandon the symbol ∞ and develop a whole collection of new symbols to describe the varying levels of infinity. The ones we use may look strange when you first meet them, but they are the ones used by Georg Cantor when he first introduced these ideas towards the end of the nineteenth century. Cantor used the first letter of the Hebrew alphabet, the 'aleph' written as $\aleph$. The smallest infinite number he denoted by $\aleph_0$ (aleph zero).

The standard set for $\aleph_0$ is the set {1, 2, 3, ….} of all positive integers. Any set that can be put into 1-1 correspondence with this set is said to have $\aleph_0$ elements. Such a set can be enumerated in a list: x_1, x_2, … and conversely any set which can be so listed has cardinal number $\aleph_0$.

Example 31: The set of even positive integers has cardinal number $\aleph_0$. The fact that we have thrown away half the numbers does not alter the fact that the even positive integers can be listed in an infinite list: 2, 4, 6, … Unlike finite sets where taking some elements out reduces the size, infinite sets can have elements removed without changing their size.

But we have yet to be convinced that there really is a bigger set than {1, 2, 3, …}.

Example 32: The set of natural numbers has size $\aleph_0$. The natural numbers are 0, 1, 2, 3, … and since these are listed in an infinite list the set of all the natural numbers has size $\aleph_0$.

But we added one extra numbers to the set of positive integers so should we not get one more? Not at all – adding one more makes no difference to the size of an infinite set. In the arithmetic of infinite cardinal numbers $\aleph_0 + 1 = \aleph_0$.

(Before you start thinking about subtracting $\aleph_0$ from both sides to get a contradiction let me hasten to point out that subtraction is not possible with infinite cardinal numbers.)

Example 33: The set of all integers has size $\aleph_0$.
Although we usually consider the integers in two infinite lists:

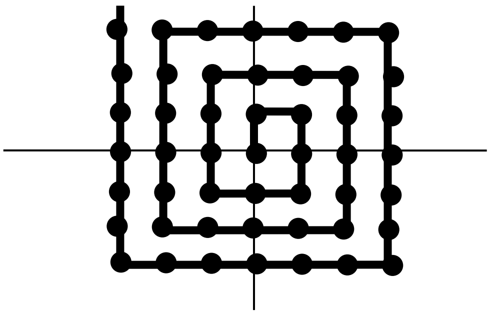
$$0,\ 1,\ 2,\ 3,\ \ldots\ldots\ldots$$
$$-1, -2, -3, -4\ \ldots\ldots\ldots$$

we can easily merge these lists to produce a single infinite list:

$$0, -1, 1, -2, 2, -3, 3, \ldots\ldots\ldots\ldots\ldots$$

So $\aleph_0 + \aleph_0 = \aleph_0$.

Example 34: $\#(\mathbb{Z} \times \mathbb{Z}) = \aleph_0$.
To see this we need to imagine the elements of $\mathbb{Z} \times \mathbb{Z}$ represented by points with integer coordinates in the *x*-*y* plane. We have $\aleph_0$ lists, each with $\aleph_0$ points. It would seem very complicated to merge all these lists. But in fact it isn't. Just start at the origin (0, 0) and trace a spiral path around the origin so that every point with integer coordinates is eventually reached. The elements of $\mathbb{Z} \times \mathbb{Z}$ can simply be listed in the order in which they occur in this spiral.



$\mathbb{Z} \times \mathbb{Z} = \{(0, 0), (0, 1), (1, 1), (1, 0), (1, -1), (0, -1), (-1, -1), (-1, 0), (-1, 1), (-1, 2), (0, 2), (1, 2), (2, 2), (2, 1), (2, 0), (2, -1), (2, -2), (1, -2), (0, -2), (-1, -2), (-2, -2), (-2, -1), \dots\}$
So $\aleph_0 \times \aleph_0 = \aleph_0$. (This means that division is not possible for infinite numbers.)

This arithmetic is beginning to seem very boring. But wait. There is a surprise just around the corner. Neither addition nor multiplication are powerful enough to break the $\aleph_0$ barrier. What about exponentiation? Before we go for $\aleph_0^{\aleph_0}$ we will just try $2^{\aleph_0}$. That will be enough.

Recall that for finite sets 2^n is the number of subsets of a set of size n. Recall too that $2^n > n$ for every finite number n. Would it not be exciting if this were the case for infinite sets as well? We would not only break the $\aleph_0$

barrier but we would suddenly get infinitely many infinite numbers: $\aleph_0 < 2^{\aleph_0} < 2^{2^{\aleph_0}} < \ldots$

If n =#S then we define $2^n = \#\wp(S)$. This can be demonstrated to be true, for finite numbers, and is used as a definition for infinite numbers.

We define $\aleph_{n+1} = 2^{\aleph_n}$ and so, once we prove that $\aleph_n < \aleph_{n+1}$ we get an infinite list of infinite numbers:

$$\aleph_0 < \aleph_1 < \aleph_2 < \ldots$$

Theorem 6: Every set has more subsets than elements.
Proof: Let S be a set and suppose that $\wp(S)$ has the same size as S. This means that there is a 1-1 and onto function from S to the set of subsets of S. Let's call this *f*.

We now define $X = \{x \in S \mid x \notin f(x)\}$, the set of all those elements of S which do not belong to the subset they correspond to. Now X is clearly a subset of S and, since *f* is onto, we must have $X = f(x)$ for some $x \in S$. The question we now ask is "Does $x \in X$?"

This is a perplexing question because it has no satisfactory answer. For if $x \in X$ then $x \in \{x \in S \mid x \notin f(x)\}$ and so $x \notin f(x)$. But $f(x) = X$ and so this gives a contradiction. But on the other hand, if $x \notin X = f(x)$ then *x* satisfies the property that defines X, which means that $x \in X$. Both alternatives lead to a contradiction yet one of them must be true. The only way out of this mess is for our original assumption to be false.

So S, and its power set $\wp(S)$, don't have the same number of elements.
But $\wp(S)$ can't be smaller because it contains the subsets $\{s\}$ of size 1 and there's one of these for every $s \in S$. So $\wp(S)$ must be strictly bigger.
Corollary: $2^n > n$ for all cardinal numbers n, finite or infinite.

We define $\aleph_{n+1} = 2^{\aleph_n}$ and so we get an infinite list of infinite numbers: $\aleph_0 < \aleph_1 < \aleph_2 < \ldots$

$\aleph_0^{\aleph_0}$ is defined as the number of functions $f:\mathbb{N}\rightarrow\mathbb{N}$. This might seem very much bigger than $2^{\aleph_0}$ but, in fact it is possible to show that $\aleph_0^{\aleph_0} = 2^{\aleph_0} = \aleph_1$.

There are deep logical mysteries awaiting anyone who wants to go further into infinite cardinal numbers (luckily we don't need to here). For a start there are not only the infinite numbers $\aleph_0$, $\aleph_1$, $\aleph_2$, …. but there are even bigger ones – bigger than any in the list. Also we have suggested that $\aleph_1$ is the next infinite number after $\aleph_0$ but we didn't prove it for the simple reason that it is unprovable. It has been proved that the question "is there a cardinal number between $\aleph_0$ and $\aleph_1$" is undecidable!

Cantor actually defined $\aleph_1$ to be the next infinite number after $\aleph_0$ so for him the question was "is $\aleph_1 = 2^{\aleph_0}$?" I have taken the view that although we can't prove there are no numbers between $\aleph_0$ and $2^{\aleph_0}$ we certainly will never be able to find one (to do so would be to contradict the undecidability) and so we may as well define $\aleph_1$ to be $2^{\aleph_0}$.

Putting those mysteries behind us let us now ask if this is this any use. The answer is definitely "yes". Often a question of existence can be answered by counting. Is there somebody at the party who wasn't invited? Well there are 20 names on the guest-list and 21 people in the room. Therefore there must exist a 'gate-crasher'.

We'll apply infinite cardinality to the question of computability. Computer programs can be written to compute functions from the positive integers to the positive integers, such as a program to compute the *n*'th prime. Is there such a function for which no computer program can ever be written? It seems rather dangerous to make the bold assertion that non-computable functions exist because who knows how clever future generations of programmers will be and how powerful their computers will be! And yet we can prove that non-computable functions do exist. Why? Simply because there are only $\aleph_0$ potential computer programs in any programming language (finitely many of each size so they can be listed according to size). Yet there are $\aleph_1$ possible functions on the set of natural numbers. Hence there are more such functions than there are potential programs and so there must exist functions that cannot be programmed.

A natural follow-up question is “well, can you show me such a function?” Unfortunately counting can’t do that, but in a later chapter we’ll exhibit a specific non-computable function.

EXERCISES FOR CHAPTER 3

EXERCISES 3A (Sets)

Ex 3A1: Let A = {1, 3, 5, 9, 11}, B = $\{n \in \mathbb{N} \mid n < 10\}$, C = $\{n \in \mathbb{N} \mid n \text{ is prime}\}$, D = A ∩ C and E = A – C. List the elements of:

(a) B;
(b) D;
(c) E;
(d) B – A;
(e) A ∪ (B ∩ C);
(f) D × E;
(g) $\wp$(D).

Ex 3A2: Simplify each of the following (each answer is either S or ∅):

(a) S ∩ S;
(b) S – S;
(c) S ∩ ∅;
(d) S × ∅;
(e) S ∪ ∅

Ex 3A3: Which of the following statements are true for all sets R, S and T?

(a) S ∩ T = T ∩ S;
(b) S – T = –(T – S);
(c) S × T = T × S;
(d) R ∩ (S ∩ T) = (R ∩ S) ∩ T;

(e) $-(S \cap T) = -S \cap -T$;
(f) $-(S \cup T) = -S \cap -T$;
(g) $R \cap (S - T) = (R \cap S) - (R \cap T)$;
(h) $R \cup (S - T) = (R \cup S) - (R \cup T)$

Ex 3A4: Let A = {3, 4, 5} and
$B = \{n \in \mathbb{N} \mid \exists q \in \mathbb{N}[(n = 2q - 1) \wedge (q < 5)]\}$.

(a)Find the elements of: $\wp(A \cap B) \times (A - B)$.

(b) How many elements are there in

$$\wp(A \times \wp(A)) \cup A?$$

(c) Find the elements of

$$[\wp(A \cap A) \times \varnothing] \cup [\wp(\varnothing) \times (A \cup \varnothing)]$$

Ex 3A5: If A = {1, 2, 3} and B = {2, 4} write down the elements of:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A \times B$;
(d) $\wp(A)$.

Ex 3A6: If A = {1, 2, 3}, B = {0, 1} and C = {0, 2, 4} write down:

(a) $A \cap (B \cup C)$;
(b) $B \times C$;
(c) $\wp(C) - \wp(A \cup B)$.

Ex 3A7: Let A = {1, 10, 101} and B = {1, 11, 011} be sets of binary strings.
Find the elements of each of the following sets:

(a) $A \cap B$;
(b) $A \cup B$;
(c) $A - B$;
(d) $A \times B$;
(e) AB;
(f) $\wp(A)$.

Exercise Set 3B (Relations)

Ex 3B1: Let C be the relation on the set of your (family) relations defined by:

xC$y \leftrightarrow x = y$ or
x is the brother of y or
x is the sister of y or
x is a (first) cousin of y.

(a) Express C in terms of the parent relation P. [Who are the links between you and your cousins?]
(b) Is C an equivalence relation?

Ex 3B2: Let R and S be relations on the set {1, 2, 3} defined as follows:
R = {(1,3), (2,1), (2,2), (2,3)}, S = {(1,1), (1,2), (2,3), (3,1)}. Find:

(a) RS;
(b) SR;
(c) R + S;
(d) R^{-1}.

Ex 3B3: A play called 'Six Degrees of Separation' is based on the relation K, defined on the set W of people alive in the world by xKy if "x knows y". The idea behind the play was that any two people in the world are related by the product K^6. Explain what you think this means.

Exercise Set 3C (Equivalence Relations)

Ex 3C1: Let R be the relation on the set of real numbers defined by xRy if $y = xq$ for some positive rational number q. Prove that R is an equivalence relation. Find an equivalence class with a finite number of elements.

Ex 3C2: Let $\approx$ be the relation on the set $\mathbb{R}$ where x $\approx$ y means "$x^m = y^n$ for some positive integers m, n". Prove that $\approx$ is an equivalence relation. Which equivalence classes are finite?

Ex 3C3: The relation R on the set {2, 3, 4, 8, 15} defined by xR$y \leftrightarrow$ '$x + y$ is not prime' is an equivalence relation. Find the equivalence classes. What about if R is a different set of positive integers?

Ex 3C4: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

(a) R on the set $\mathbb{N}$ where xR$y \leftrightarrow x \geq y$;

(b) S on the set $\mathbb{N}$ where xSy $\leftrightarrow x^2 = y^2$;

(c) T on the set $\mathbb{R}^+$ where xR$y \leftrightarrow$ log (y/x) is an integer.

Ex 3C5: Let R be the relation on $\mathbb{Z}$ defined by $x\mathrm{R}y$ if $x + y$ is even. Prove that R is an equivalence relation.

Ex 3C6: For each of the following relations determine which of the reflexive, symmetric and transitive properties hold. Which, if any, are equivalence relations?

(a) R on the set of people where $x\mathrm{R}y$ means 'x is the brother of y';

(b) S on the set $\mathbb{Q}$ where $x\mathrm{S}y \leftrightarrow y = x.2^n$ for some $n \in \mathbb{Z}$.

Ex 3C7: Define the relation R on the set
S = {1, 2, 3, ..., 100} by:

$$a\mathrm{R}b \leftrightarrow a \mid 2^r b \text{ for some } r \in \mathbb{N} \text{ and } b \mid 2^s a \text{ for some } s \in \mathbb{N}.$$

(a) Prove that R is an equivalence relation.

(b) Find the number of equivalence classes.

Ex 3C8: Suppose that the relation $\approx$ is defined on the set of all binary strings by:
A $\approx$ B if there exist strings X, Y such that AX = XB and YA = BY.

(a) Prove that $\approx$ is an equivalence relation.

(b) Find the equivalence class containing 11001.

Ex 3C9: Let R be the relation on $\mathbb{N}$ defined by $x\mathrm{R}y$ if $3x + 4y$ is a multiple of 7.
Prove that R is an equivalence relation.

Ex 3C10: Let R, S, T be the following relations on ℕ: xRy means $x < y$; xSy means $x < y + 1$;
T means $x < 2y$.

(a) Prove that $R^2 = S$.

(b) Prove that RT ≠ TR. Show this by finding two numbers m, n such that mRTn is TRUE but mTRn is FALSE.

Ex 3C11: Define the relation R on the set of all binary strings by αRβ if αβ has even parity (an even number of 1's). Prove that R is an equivalence relation and find the shortest string in the equivalence class containing the string 1011.

Exercise Set 3D (Functions)

Ex 3D1: Let S be a finite set of size n. Find, as a power of 2, the number of functions from
S × S to $\wp$(S). How many of these are onto?

Ex 3D2: Let S = {1, 2} and T = {1, 2, 3}. Are there more functions from S to T or functions from T to S? What about 1-1 functions? What about onto functions?

Ex 3D3: Let L be the set of lotteries conducted in N.S.W. during a given year and let N be the set of all the numbers of the tickets. Define w:L→N by $w(x)$ = number of the ticket winning 1st prize in lottery x. Which do you think is more likely, for w to be 1-1 or for it to be onto? Discuss.

Ex 3D4: Let S = {1, 2}. Find the number of functions from $\wp(S) \cup S$ to $\wp(S) \times S$. How many of these are 1-1? How many are onto? [**NOTE:** The elements of S are disjoint from the subsets of S. We distinguish between a subset with one element and that element itself.]

SOLUTIONS FOR CHAPTER 3

Ex 3A1: (a) 0, 1, 2, 3, 4, 5, 6, 7, 8, 9;
(b) 3, 5, 11;
(c) 1, 9;
(d) 0, 2, 4, 6, 7, 8;
(e) 1, 2, 3, 5, 7, 9, 11;
(f) (3,1), (3,9), (5,1), (5,9), (11,1), (11,9);
(g) $\emptyset$, {3}, {5}, {11}, {3,5}, {3,11}, {5,11}, {3,5,11}.

Ex 3A2: For (a) and (e) the answer is S, for (b), (c) and (d) the answer is $\emptyset$.

Ex 3A3: (a), (d), (f), (g) are TRUE. The others are FALSE.

Ex 3A4: (a) $A = \{3, 4, 5\}$; $B = \{1, 3, 5, 7\}$;
$A \cap B = \{3, 5\}$;
$A - B = \{4\}$;
$\wp(A \cap B) = \{\emptyset, \{3\}, \{5\}, \{3, 5\}\}$;
$\wp(A \cap B) \times (A - B) = \{(\emptyset, 4), (\{3\}, 4), (\{5\}, 4), (\{3, 5\}, 4)\}$

(b) $|A| = 3$;
$|\wp(A)| = 2^3$;
$|A \times \wp(A)| = 3 \times 2^3 = 24$;
$|\wp(A \times \wp(A))| = 2^{24}$;

Since A and $\wp(A \times \wp(A))$ are disjoint,

$$|\wp(A \times \wp(A)) \cup A| = 2^{24} + 3.$$

(c) $\wp(A \cap A) \times \varnothing = \varnothing$ so the set can be simplified to $\wp(\varnothing) \times (A \cup \varnothing)$. As $\wp(\varnothing) = \{\varnothing\}$, the set can be simplified to $\{\varnothing\} \times A = \{(\varnothing, 3), (\varnothing, 4), (\varnothing, 5), (\varnothing, 6)\}$.

Ex 3A5: (a) $A \cap B = \{2\}$;

(b) $A \cup B = \{1, 2, 3, 4\}$;

(c) $A \times B = \{(1, 2), (1, 4), (2, 2), (2, 4), (3, 2), (3, 4)\}$;

(d) $\wp(A) = \{\varnothing, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$.

Ex 3A6: (a) $A \cap (B \cup C) = \{1, 2, 3\} \cap \{0, 1, 2, 4\} = \{1, 2\}$;

(b) $B \times C = \{(0, 0), (0, 2), (0, 4), (1, 0), (1, 2), (1, 4)\}$;

(c) $\wp(C) - \wp(A \cup B) = \{\{4\}, \{0, 4\}, \{2, 4\}, \{0, 2, 4\}\}$.

Ex 3A7:

(a) {1};

(b) {1, 10, 11, 011, 101};

(c) {10, 101};

(d) {(1, 1), (1, 11), (1, 011), (10, 1), (10, 11), (10, 011), (101, 1), (101, 11), (101, 011);

(e) {11, 111, 1011, 101, 10011, 10111, 101011};

(f) {∅, {1}, {01}, {101}, {1, 10}, {1, 101}, {10, 101}, {1, 10, 101}}.

Ex 3B1: One's cousins, brothers and sisters and oneself are those who are grandchildren of one's grandparent, so $C = P^2P^{-2}$ where P^{-2} means $(P^{-1})^2 = (P^2)^{-1}$ = 'grandchild of'.

NOTE: normal algebraic cancellation doesn't apply!

Ex 3B2: (a) {(1, 1), (2, 1), (2, 2), (2, 3)};
(b) {(1, 3), (1, 1), (1, 2), (3, 3)};
(c) {1, 1), (1, 2), (1, 3), (2, 2), (2, 3), (3, 1)};
(d) (1, 2), (2, 2), (3, 1), (3, 2)}

Ex 3B3: Given any two people P and Q in the world there exist 5 people P_1, P_2, P_3, P_4, P_5 such that P knows P_1, P_1 knows P_2, P_2 knows P_3, P_3 knows P_4 knows P_5, and P_5 knows Q.

[The chain may be shorter in some cases, but note that everybody knows themselves, so a shorter chain can be extended to one of length 6.]

Ex 3C1: *Reflexive:* $x = x1$ and 1 is rational.
Symmetric: Suppose $x\text{R}y$. Then $y = xq$ for some $q \in \mathbb{Q}^+$ and so $x = yq^{-1}$ and since
$q^{-1} \in \mathbb{Q}^+$, $y\text{R}x$.
Transitive: Suppose $x\text{R}y$ and $y\text{R}z$. Then $y = xq$ and $z = yr$ for some $q, r \in \mathbb{Q}^+$.

Then z = $(xq)r = x(qr)$. Since $qr \in \mathbb{Q}^+$, xRz.
{0} is the only finite equivalence class.

Ex 3C2: *Reflexive:* $x^1 = x^1$ so $x \approx x$ for all x.
Symmetric: Suppose $x \approx y$. Thus, $x^m = y^n$ for some positive integers $m, n \in \mathbb{Z}^+$.
Then since $y^n = x^m$, y ≈ x.
Transitive: Suppose $x \approx y$ and $y \approx$ z. Thus $x^m = y^n$ and $y^s = z^t$ for some $m, n, s, t \in \mathbb{Z}^+$.
Then $xx^{ms} = (x^m)^s = (y^n)^s = (y^s)^n = z^{nt}$. Hence $x \approx$ z.
The only finite equivalence classes are {0} and {1, –1}. All others are infinite.
eg [2] = $\{2^{\pm n} \mid n \in \mathbb{Z}\}$.

Ex 3C3: {2, 4, 8}, {3, 15}.

Ex 3C4: (a) reflexive and transitive;
(b) reflexive, symmetric and transitive;
(c) none;
(d) reflexive, symmetric and transitive.
(b) and (d) are equivalence relations.

Ex 3C5: *Reflexive:* $x + x = 2x$ is even for all integers x;
Symmetric: if $x + y$ is even then $y + x$ is even;
Transitive: Suppose $x + y$ and y + z are even.
Then $(x + y) + (y + z) = x + z + 2y$ is even.
Hence x + z is even.

Ex 3C6: (a) Symmetric only [It is not reflexive since I'm not my own brother. It is not transitive: e.g. take x = z = you and y = your brother.]

(b) Reflexive, symmetric and transitive (i.e. an equivalence relation).

Ex 3C7:

(a) *Reflexive:* Let $a \in$ S. The since $a \mid 2a$, aRa.
Hence R is reflexive.
Symmetric: R is clearly symmetric as defined.
Transitive: Suppose aRb and bRc.
Then $a \mid 2^r b,\ b \mid 2^s a,\ b \mid 2^u c,\ c \mid 2^v b$ for some $r, s, u, v \in \mathbb{N}$.
Hence $a \mid 2^{r+u}c$ and $c \mid 2^{v+s}a$.
Since $r + u$ and $v + s$ are both natural numbers it follows that aRc.
Hence R is transitive.
Thus we have shown that R is an equivalence relation.
(b) Two numbers are equivalent under R if the are equal, give or take some factors of 2. Thus each equivalence class contains exactly one odd number and so the number of equivalence classes is exactly 50.

Ex 3C8: (a) *Reflexive:* Let A be a binary string.
Since Aλ = λA we have A $\approx$ A.
Symmetric: Suppose A $\approx$ B. Then for some strings X, Y, AX = XB and YA = BY.
Hence BY = YA and XB = AX whence B $\approx$ A.

Transitive: Suppose A ≈ B and B ≈ C. Then for some strings X, Y, S, T:

AX = XB and YA = BY and BS = SC and TB = CT.

Hence A(XY) = XBY = (XY)A and (TY)A = TBY

= C(TY) and so A ≈ C.

(b) The equivalence class containing 1011 is {1011, 0111, 1110, 1101}. Note that these are just the cyclic rearrangements of 1011.

Ex 3C9: *Reflexive:* Let $x \in \mathbb{N}$. Then $3x + 4x = 7x$ is a multiple of 7 and so xRx.

Symmetric: Suppose xRy. Then $7 \mid 3x + 4y$.

Thus $7 \mid 7(x + y) - (3x + 4y) = 4x + 3y$. Hence yRx.

Transitive: Suppose xRy and yRz. Then $7 \mid 3x + 4y$ and $7 \mid 3y + 4z$.

Hence $7 \mid (3x + 4y) + (3y + 4z) = 3x + 7y + 4z$ and so $7 \mid 3x + 4z$. Hence xRz.

Ex 3C10: (a) Suppose xR^2y. Then for some z, xRz and zRy.

Thus $x < z$ and $z < y$, and so $z \le y + 1$ and hence $x < y + 1$. Thus xSy.

Conversely suppose xSy. Then $x < y + 1$.

Let $z = y$. Then $x < z$ and $z < y + 1$.

So xRz and zRy and hence xR^2y.

(b) 2RT2, since 2R3 and 3T2 (i.e. 3 < 2.2).

But 2TR2 is FALSE. For suppose there was a number z with 2Tz and zR2. Then 2 < 2z and z < 2. Thus 1 < z < 2 which is impossible for natural numbers.

Ex 3C11: *Reflexive:* $\alpha\alpha$ has twice the number of 1's as α and so has even parity.
Symmetric: Although $\alpha\beta \neq \beta\alpha$ they contain the same number of each symbol.
Therefore if $\alpha\beta$ has even parity, so does $\beta\alpha$.
Transitive: Suppose $\alpha R\beta$ and $\beta R\gamma$. Then $\alpha\beta$ and $\beta\gamma$ each has even parity.
Thus $\alpha\beta\beta\gamma$ has even parity. Removing the $\beta\beta$ takes out an even number of 1's and so $\alpha\gamma$ has even parity, i.e. $\alpha R\gamma$.
The equivalence class containing 1011 consists of all strings of odd parity and the shortest such string is 1.

Ex 3D1:
There are $(2^n)^{n^2} = \mathbf{2^{n^3}}$ functions from $S \times S$ to $\wp(S)$.
$|S \times S| = n^2$, while $|\wp(S)| = 2^n$.
For all $n \geq 5$, 2^n is bigger than n^2 and in these cases there can be no onto functions from the smaller $S \times S$ to the larger $\wp(S)$. This is also the case for $n = 1$.

For $n = 2$, both sets have the same size viz. 4. Onto functions from a finite set to one with the same number of elements cannot 'double up' and so must be 1-1 (and conversely). So in this case the number of 1-1 functions is the number of 1-1 and onto functions, that is the number of permutations on 4 elements. The answer for $n = 2$ is thus $4! = 24$.

Similarly for $n = 4$ (where both sets have the same size of 16) the number of onto functions is 16!

For $n = 3$ the situation is a little trickier. We have to count the number of onto functions from a set of size 9 to a set of size 8. Such an onto function must involve 7 of the 9 elements of $S \times S$ having an exclusive image (no other element being mapped to the same image) with the remaining two elements of $S \times S$ mapping together to the remaining element of $\wp(S)$.
There are 8 choices for the one element of $\wp(S)$ which has two elements of $S \times S$ mapping to it. For each of these, there are (9.8)/2 choices for the two elements of $S \times S$ to map to it. For each of these 8.9.4 ways of arranging the doubling up, there are 7! ways of mapping the remaining 7 elements to their unique images. Hence the number is 4.7! = 20160.
Thus if $O(n)$ is the number of onto functions:

$$O(n) = 0 \text{ if } n = 1 \text{ or } n \geq 5;$$
$$O(2) = 24;$$
$$O(3) = 4.7!;$$
$$O(4) = 16!$$

Ex 3D2: There are more functions from S to T than from T to S, more 1-1 functions from S to T than from T to S but more onto functions from T to S than from S to T.

The actual numbers are:

	S→T	T→S
all functions	9	8
1-1	6	0
onto	0	6

Ex 3D3: The statement '*w* is 1-1' means that nobody wins twice. While this is not certain it's a highly likely event. The statement "*w* is onto" means that every ticket number will win 1st prize at least once during the year – an impossible event since quite clearly there are more tickets in each lottery than there are lotteries during the year.

Ex 3D4: |S| = 2. Now $\wp$(S) and S have no elements in common (we distinguish between {1} and 1 itself) so $|\wp(S) \cup S| = |\wp(S)| + |S| = 2^2 + 2 = 6$, while $|\wp(S) \times S| = 2^2.2 = 8$.
The number of functions from $\wp(S) \cup S)$ to $\wp(S) \times S$ is thus $8^6 = 262144$.
The number which are onto is zero (you can't have an onto function from a smaller to a larger set). The number which are 1-1 is 8.7.6.5.4.3 = 40320.