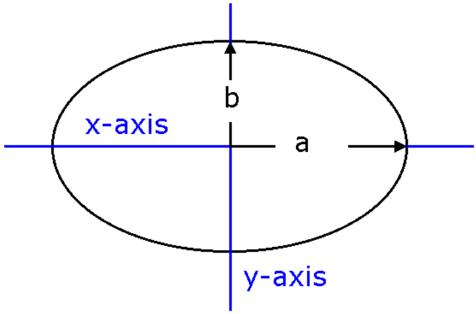
5. CONICS AND QUADRIC SURFACES

§5.1. Conics

The equation of a circle, centred at the origin and with radius r, is $x^2 + y^2 = r^2$.

An ellipse centred on the origin, and with its axes along the x- and y-axes, has an equation of the form:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



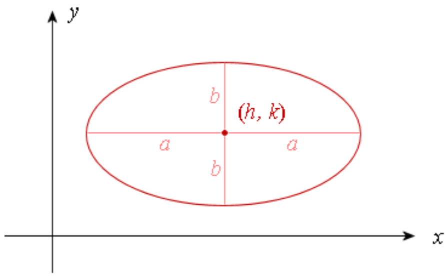
Here, a and b are the semi major and axes.

Both of these have the form $Ax^2 + By^2 = 1$. Once we move the centre we introduce x and y terms. For example, if the above ellipse is moved to the point (h, k) its equation becomes

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1.$$

This has the form:

$Ax^2 + By^2 + Cx + Dy = 1.$

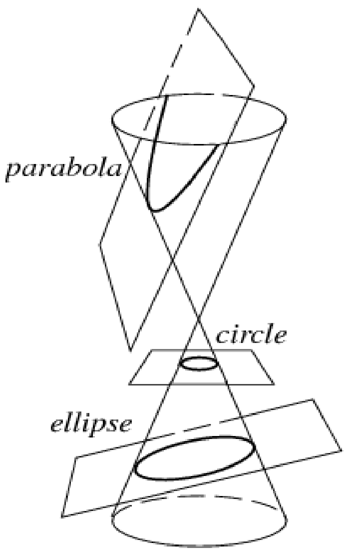


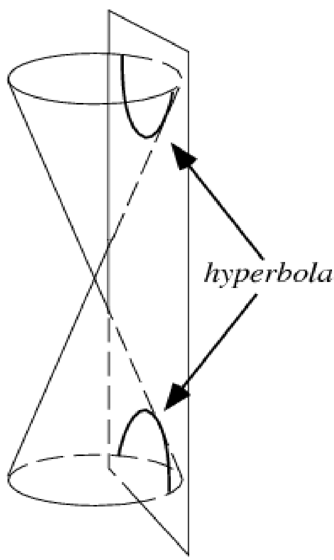
Once we start rotating the axes we introduce *xy* terms.

The standard rectangular hyperbola has equation $xy = 1$, but if we rotate it through 45° so that the axes of symmetry lie along the coordinate axes its equation becomes:

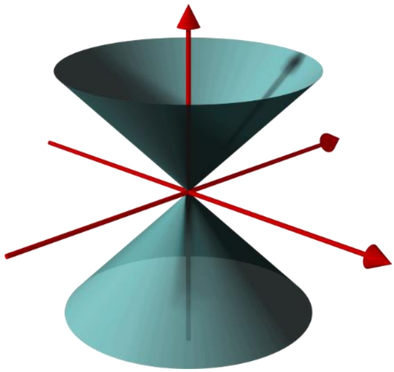
$$x^2 - y^2 = 2.$$

These are examples of conics. The geometrical definition of a **conic** is that it is the intersection of a cone with a plane. Traditionally a cone has a base, but the cone we're thinking of here is what you might call a 'double cone'. If you rotate a line through another line that intersects the original line, the infinite surface that results is a **cone**.





A conic is usually what we would call a 'curve' but it's possible, by taking the intersecting plane through the vertex of the cone, to get a pair of straight lines, or even a single point. These are called **degenerate conics**.



Example 1: The cone $x^2 + y^2 = z^2$ has its vertex at the origin and its axis of rotational symmetry along the z-axis. If we intersect this with the *x*-*y* plane we get just the origin. If we intersect it with the *x*-z plane we get a pair of straight lines.

§5.2. Degenerate Conics

The word 'degenerate' in mathematics refers to cases that technically satisfy the definition, but which are uninteresting because they are much simpler than the usual examples. Another word that is used in this sense is 'trivial', as in the trivial solution to a system of homogeneous linear equations.

Other degenerate conics arise from a degenerate cone, namely a cylinder. A cylinder, after all, is the surface obtained when one line rotates around a parallel one. It might be argued that these lines don't intersect. We certainly don't want to include the case of two *skew* lines intersecting, but we sometimes consider two parallel lines as intersecting at a 'point at infinity'. In order to make this geometric definition agree with the algebraic definition that we'll shortly present, we must include a cylinder as an honorary cone.

Degenerate conics include, as we have seen, a pair of intersecting straight lines. Intersecting a cylinder with a plane we get circles and ellipses (these are certainly not degenerate but we can get them from an actual cone), a pair of parallel lines, a single line and the empty set. These last three are considered degenerate.

Example 2: Intersecting the cylinder $x^2 + y^2 = 1$ with the x-z plane we get a pair of parallel lines, at a distance of 2 apart. In the x-y plane we can get such a pair of parallel lines from the equation $x^2 - 1 = 0$.

Example 3: Intersecting the cylinder $x^2 + y^2 = 1$ with the plane $x = 1$ (a tangent plane to the cylinder) we get a single line.. In the x-y plane we can get a single line from the equation $x^2 - 2x + 1 = 0$.. Since this is $(x - 1)^2 = 0$ the line is the one with the simpler equation $x = 1$.

Example 4: Intersecting the cylinder $x^2 + y^2 = 1$ with plane $x = 2$ we get the empty set. In the x-y plane we can get the empty set from the equation $x^2 + 1 = 0$.

§5.3. The Algebraic Definition of a Conic

The algebraic definition of a **conic** is that it is the set of points that satisfy an equation of the form:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$$

where at least one of a, b and h is non-zero.

It can be shown that the two definitions agree, provided we allow the cylinder to be considered as a degenerate cone. However the only extra shapes that arise by including the cylinder are degenerate ones.

Throughout this chapter we'll only be considering vectors in $\mathbb{R}^3$, where the distinction between vectors and scalars can be maintained. So we'll revert to the practice of printing vectors in bold type.

Consider the conic $ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$.

If we let $\mathbf{v} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ and $Q = \begin{pmatrix} a & h & g \\ h & b & f \\ g & f & c \end{pmatrix}$ then we can express the equation as $\mathbf{v}^{\mathbf{T}}Q\mathbf{v} = 0$.

Strictly speaking it's the1×1 matrix whose single component is this expression, but we identify 1×1 matrices with their associated scalar.

In most cases, as we show in the next theorem, we can translate the conic so as to eliminate the x and y terms to obtain an equation of the form $ax^2 + by^2 + 2hxy = c$. Moreover, if $c \neq 0$ we can divide through by c and obtain an equation of the form:

$$ax^2 + by^2 + 2hxy = 1.$$

So, apart from those cases where translation cannot remove the x and y terms, a conic has one or other of the forms:

$$ax^2 + by^2 + 2hxy = 1 \text{ or}$$
$$ax^2 + by^2 + 2hxy = 0.$$

If we now put $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Q = \begin{pmatrix} a & h \\ h & b \end{pmatrix}$ we can write these equations as either $\mathbf{v}^T Q\mathbf{v} = 1$ or $\mathbf{v}^T Q\mathbf{v} = 0$.

Theorem 1: If a conic has the equation:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$$

and $h^2 \neq ab$ then after a suitable translation the equation becomes:

$$aX^2 + bY^2 + 2hXY = K, \text{ for some K.}$$

Proof: Suppose $h^2 \neq ab$. Let $\mathrm{X} = x - p$ and $\mathrm{Y} = y - q$ for some p, q.
The lines $x = p$ and $y = q$ become the new axes and (p, q) gets translated to the origin.
Then $a(\mathrm{X} + p)^2 + b(\mathrm{Y} + q)^2 + 2g(\mathrm{X} + p) + 2f(\mathrm{Y} + q)$
$+ 2h(\mathrm{X} + p)(\mathrm{Y} + q) + c = 0$.
Hence:
$a\mathrm{X}^2 + b\mathrm{Y}^2 + 2h\mathrm{XY} + 2(ap + g + hq)\mathrm{X} + 2(bq + f + hp)\mathrm{Y}$
$+ (2p^2 + 2q^2 + 2gp + 2fq + 2hpq + c) = 0$.
Since $h^2 \neq ab$ we can solve the equation:

$$\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = -\begin{pmatrix} g \\ f \end{pmatrix}.$$

Using these values of p, q we get, as the equation of the translated conic:

$$a\mathrm{X}^2 + \mathrm{bY}^2 + 2h\mathrm{X}Y = \mathrm{K}$$

where $\mathrm{K} = -(2p^2 + 2q^2 + 2gp + 2fq + 2hpq + c)$. ✋☺

Example 5: Find a translation of the conic

$$3x^2 + y^2 + 2xy + 10x - 6y + 7 = 0$$

so that it has an equation with no x and y terms.
Solution: Translate (p, q) to the origin.
Then $\mathrm{X} = x - p$, $\mathrm{Y} = y - q$.
Hence $3(\mathrm{X} + p)^2 + (\mathrm{Y} + q)^2 + 2(\mathrm{X} + p)(\mathrm{Y} + q)$
$+ 10(\mathrm{X} + p) - 6(\mathrm{Y} + q) + 7 = 0$, and so
$3\mathrm{X}^2 + \mathrm{Y}^2 + 2\mathrm{XY} + 2(3p + q + 5)\mathrm{X} + 2(q + p - 3)\mathrm{Y}$
$+ (3p^2 + q^2 + 2pq + 10p - 6q + 7) = 0$.

Solving $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \end{pmatrix}$ we get:

$$\begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}^{-1}\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 3 \end{pmatrix}\begin{pmatrix} -5 \\ 3 \end{pmatrix} = \begin{pmatrix} -4 \\ 7 \end{pmatrix}.$$

So, translating (–4, 7) to the origin the equation becomes:

$$3X^2 + Y^2 + 2XY = 34.$$

If $ab \neq h^2$ the general conic:

$$ax^2 + by^2 + 2gx + 2fy + 2hxy + c = 0$$

can be translated so that it has an equation of the form:

$$aX^2 + bY^2 + 2hXY = K.$$

If $K \neq 0$ we can divide through by K and so put the equation in the form:

$$aX^2 + bY^2 + 2hXY = 1.$$

The expression $aX^2 + bY^2 + 2hXY$ is called a quadratic form.

§5.4. Quadratic Forms

A **quadratic** form is an expression of the form $\mathbf{v}^T Q\mathbf{v}$, where $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{pmatrix}$ and Q is an $n \times n$ real symmetric matrix.

Example 6: A quadratic form in two variables x, y has the form $ax^2 + by^2 + 2hxy = (x, y)\begin{pmatrix} a & h \\ h & b \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.

Example 7: The expression:

$$5x^2 + y^2 - 2z^2 + 4xy - 2xz + 10yz$$

is a quadratic form in 3 variables. It can be expressed as

$$(x, y, z)\begin{pmatrix} 5 & 2 & -1 \\ 2 & 1 & 5 \\ -1 & 5 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

Theorem 2: If Q is an $n \times n$ real symmetric matrix and $\mathbf{v} = \begin{pmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{pmatrix}$ then there exists an orthogonal matrix R, with $|R| = 1$, and a real diagonal matrix D, such that if $\mathbf{u} = R\mathbf{v}$ then $\mathbf{v}^T Q\mathbf{v} = \mathbf{u}^T D\mathbf{u} = \lambda_1^2 + \lambda_2^2 + \dots + \lambda_n^2$ for some real numbers $\lambda_1, \lambda_2, \dots, \lambda_n$.

Proof: Let Q be an n × n real symmetric matrix. Then its eigenvalues, $\lambda_1, \lambda_2, \dots, \lambda_n$ are real. There's a real orthogonal eigenmatrix S such that $Q = SDS^{-1} = SDS^T$. Since $|S| = \pm 1$ we may swap two columns of S to ensure that $|S| = 1$.

Let R = S^T. It is also orthogonal and $|R| = 1$.
Let $\mathbf{u} = R\mathbf{v} = S^T\mathbf{v} = S^{-1}\mathbf{v}$. Then $\mathbf{v} = S\mathbf{u}$.
So we have
$$\begin{aligned}\mathbf{v}^TQ\mathbf{v} &= (S\mathbf{u})^TQ(S\mathbf{u})\\ &= \mathbf{u}^TS^TQS\mathbf{u}\\ &= \mathbf{u}^TS^T(SDS^T)S\mathbf{u}\\ &= \mathbf{u}^T(S^TS)D(S^TS)\mathbf{u}\\ &= \mathbf{u}^TD\mathbf{u}.\\ &= \lambda_1^2 + \lambda_2^2 + \ldots + \lambda_n^2.\end{aligned}$$
✋☺

Example 8: Consider the conic $6x^2 + 3y^2 + 4xy = 1$.

Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $Q = \begin{pmatrix} 6 & 2 \\ 2 & 3 \end{pmatrix}$.

tr(Q) = 9 and |Q| = 14 so
$\chi_Q(\lambda) = \lambda^2 - 9\lambda + 14 = (\lambda - 7)(\lambda - 2)$.

The eigenvalues of Q are 2, 7.

λ = 2: $Q - 2I = \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & 1 \\ 0 & 0 \end{pmatrix}$

so $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ is an eigenvector.

λ = 7: $Q - 7I = \begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \to \begin{pmatrix} 1 & -2 \\ 0 & 0 \end{pmatrix}$

so $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector.

These are orthogonal, but don't have unit length. Instead take $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\-2\end{pmatrix}$ and $\frac{1}{\sqrt{5}}\begin{pmatrix}2\\1\end{pmatrix}$.

So S = $\frac{1}{\sqrt{5}}\begin{pmatrix}1 & 2\\-2 & 1\end{pmatrix}$ is an orthonormal eigenmatrix.

Note that |S| = 1 so we don't have to swap columns.

Hence $Q = SDS^T = R^TDR$ where $R = S^T$ and $D = \begin{pmatrix}2 & 0\\0 & 7\end{pmatrix}$.

If $\mathbf{u} = R\mathbf{v}$, then

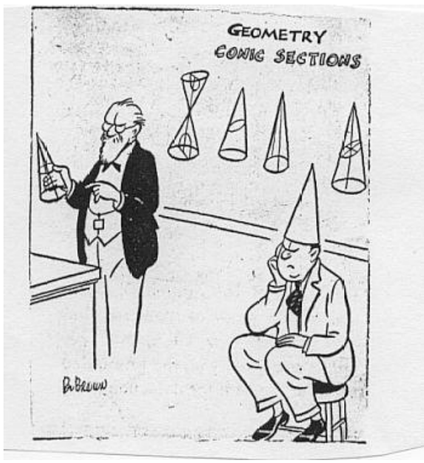
$\mathbf{v}^TQ\mathbf{v} = \mathbf{v}^TR^TDR\mathbf{v}$

$= \mathbf{u}^TD\mathbf{u}$.

If $\mathbf{u} = \begin{pmatrix}X\\Y\end{pmatrix}$ then $\mathbf{v}^TQ\mathbf{v} =$

$(X, Y)\begin{pmatrix}2 & 0\\0 & 7\end{pmatrix}\begin{pmatrix}X\\Y\end{pmatrix}$ so the equation becomes:

$2X^2 + 7Y^2 = 1$, which is the equation of an ellipse.



Recall that a 2 × 2 orthogonal matrix with determinant 1 has the form $\begin{pmatrix}\cos\theta & -\sin\theta\\\sin\theta & \cos\theta\end{pmatrix}$ and so corresponds to a rotation through θ. A similar result holds for a 3 × 3 orthogonal matrix, although we don't bother to identify precisely which rotation it corresponds to.

Theorem 3: A 3×3 orthogonal matrix with determinant 1 is the matrix of a rotation.
Proof: Suppose R is a 3×3 orthogonal matrix with $|R| = 1$ and let $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be the standard basis vectors.

Then $R\mathbf{e}_1$, $R\mathbf{e}_2$, $R\mathbf{e}_3$ are mutually orthogonal unit vectors.

Since $|R| = 1$ they will be a positive triple, like the standard basis.

Hence the three standard basis vectors can be rotated to the new basis $R\mathbf{e}_1$, $R\mathbf{e}_2$, $R\mathbf{e}_3$. ✋☺

It follows that a conic of the form $ax^2 + bx^2 + 2hxy = 1$ can be rotated so that its equation has the form:

$$Ax^2 + By^2 = 1.$$

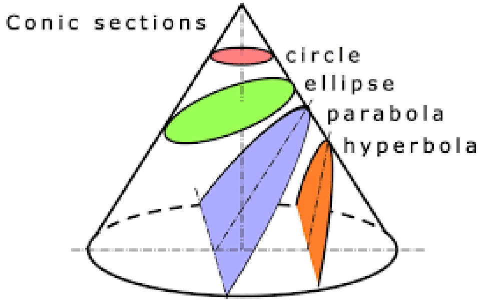
Similarly, if a conic has the equation $Ax^2 + By^2 = 0$ it can be rotated so that its equation has the form $Ax^2 + By^2 = 0$.

Here A, B will be the eigenvalues of the matrix $\begin{pmatrix} a & h \\ h & b \end{pmatrix}$.

This results in the following types of conic.
[Here, the symbols have a different meaning to the coefficients above, but these are the traditional symbols used for conics.]

CONIC $Ax^2 + By^2 = K$				
K	**A**	**B**	**Usual Equation**	**Description**
1	+ve	+ve	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$	ellipse
1	+ve	–ve	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	hyperbola
1	–ve	+ve	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$	
1	–ve	–ve	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -1$	empty set
1	+ve	0	$x^2 = a^2$	pair of parallel lines
1	–ve	0	$x^2 = -a^2$	empty set
1	0	0	$0 = 1$	empty set
1	0	+ve	$y^2 = b^2$	pair of parallel lines
1	0	–ve	$y^2 = -b^2$	empty set
0	+ve	+ve	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0$	origin
0	–ve	–ve		
0	+ve	–ve	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$	pair of intersecting lines
0	–ve	+ve		
0	+ve	0	$x = 0$	line
0	–ve	0		
0	0	0	$0 = 1$	empty set
0	0	+ve	$y = 0$	line
0	0	–ve		

You will notice that the parabola is missing from this list. That’s because there is no translation of the equation that will eliminate the *x*, *y* terms. A further analysis will show that this is the only additional conic.



§5.5. Quadric Surfaces

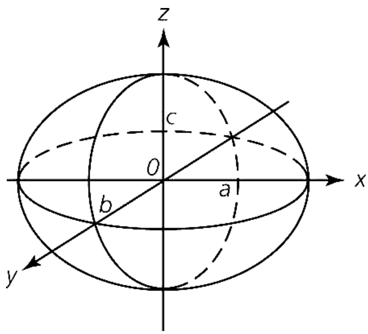
The 3 dimensional analogues of conics are **quadric surfaces**. These have an equation of the form $\mathbf{v}^T Q \mathbf{v} = \mathbf{K}$ where

$\mathbf{v} = \begin{pmatrix} x \\ y \\ z \\ 1 \end{pmatrix}$ and Q is a real symmetric matrix.

They include the **ellipsoid**. If the centre is the origin and the axes are aligned with the coordinate axes, it has an equation of the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Other familiar quadric surfaces are the (infinite) **cylinder** $x^2 + y^2 = r^2$ and the (double) **cone** $x^2 + y^2 = a^2z^2$. The parabolic mirror is a **paraboloid**, and this is another example of a quadric surface. As with conics, if we're given the equation of a quadric surface we want to be able to identify the type of surface, where it is located, and how it's oriented relative to the axes.



It's possible to analyse quadric surfaces in the same way as conics. Most quadric surfaces can be translated so that the x, y and z terms disappear. (A notable exception is the paraboloid.) The resulting equation will have the form:

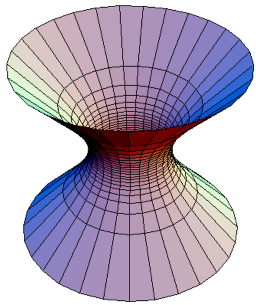
$$ax^2 + by^2 + cz^2 + 2gxy + 2hxz + 2kyz = 0 \text{ or } 1.$$

This can be written as $(x, y, z)\begin{pmatrix} a & g & h \\ g & b & k \\ h & k & c \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0$ or 1.

The left hand side of this equation is a quadratic form and can be expressed as $\mathbf{v}^T Q\mathbf{v}$ where $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. Because Q is real and symmetric it has an orthogonal basis of eigenvectors and its eigenvalues are real.

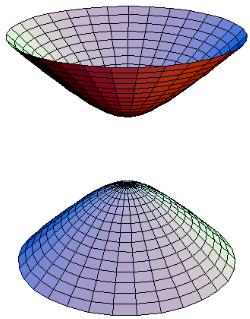
By Theorem 2 a quadratic surface with equation $\mathbf{v}^T Q\mathbf{v} = 1$ can be rotated about the origin so that it has an equation of the form $Ax^2 + By^2 + Cz^2 = 1$.

If A, B, C are all positive we get an ellipsoid with an equation which is normally written as $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$. The constants a, b and c are the lengths of the semi-axes.



If A, B are positive and C is negative we get an equation of the form: $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$, which is called a **hyperboloid with one sheet**.

If A is positive and B, C are negative we get $\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$ which is called a **hyperboloid of two sheets**. The word ‘sheet’ here refers to the separate pieces. Of course if A, B, C are all negative we get the empty set!



If one coefficient is zero we can get an elliptical prism or a hyperbolic cylinder. These are like a cylinder but where the cross sections are copies of the same ellipse or hyperbola.

Analysing quadric surfaces with an equation of the form $Ax^2 + By^2 + Cz^2 = 0$ we get a cone if A, B and C don't all have the same sign. Of course, if they have the same sign we get just the origin. Further degenerate quadric surfaces can be obtained by considering the case where one or more of A, B, C is zero.

NON-DEGENERATE QUADRIC SURFACES WITH EQUATION $Ax^2 + By^2 + Cz^2 = 1$		
# +ve coeffs	**# –ve coeffs**	**Surface**
3	0	ellipsoid
2	1	hyperboloid with one sheet
2	0	elliptical cylinder
1	2	hyperboloid with two sheets
1	1	hyperbolic cylinder
1	1	two parallel planes
0	3	empty set
0	2	
0	1	
0	0	

Example 9: Translate the quadric surface

$$2x^2 - y^2 + 3z^2 + 8xy - 2yz + 6x + 10z = 0$$

so as to express it in the form:

$$ax^2 + by^2 + cz^2 + 2gxy + 2hxz + 2kyz = K.$$

Solution: Suppose (p, q, r) is translated to the origin. Then the new coordinates are (X, Y, Z) where:

$$X = x - p,\ Y = y - q,\ Z = z - r.$$

Then $2(X + p)^2 - (Y + q)^2 + 3(Z + r)^2 + 8(X + p)(Y + q)$

$$- 2(Y + q)(Z + r) + 6(X + p) + 10(Z + r) = 0.$$

Hence $2X^2 + 4pX + 2p^2 - Y^2 - 2qY - q^2 + 3Z^2 + 6rZ$

$+ 3r^2 + 8XY + 8qX + 8pY + 8pq - 2YZ - 2rY$

$- 2qZ + 6X + 6p + 10Z + 10r = 0$

So, $2X^2 - Y^2 + 3Z^2 + 8XY - 2YZ + 2(2p + 4q + 3)X$

$+ 2(- q + 4p - r)Y + 2(2r - q + 5)Z$

$= -2p^2 + q^2 - 3r^2 - 8pq - 6p - 10r$

We solve $\begin{pmatrix} 2 & 4 & 0 \\ 4 & -1 & -1 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} p \\ q \\ r \end{pmatrix} = \begin{pmatrix} -3 \\ 6 \\ -5 \end{pmatrix}$.

$$\left(\begin{array}{ccc|c} 2 & 4 & 0 & -3 \\ 4 & -1 & -1 & 6 \\ 0 & -1 & 2 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 4 & 0 & -3 \\ 0 & -9 & -1 & 12 \\ 0 & -1 & 2 & -5 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 4 & 0 & -3 \\ 0 & 1 & -2 & 5 \\ 0 & -9 & -1 & 12 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|c} 2 & 4 & 0 & -3 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & -19 & 57 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 4 & 0 & -3 \\ 0 & 1 & -2 & 5 \\ 0 & 0 & 1 & -3 \end{array}\right).$$

So $r = -3$, $q = -1$, $p = ½$.
Translating (½, –1, –3) to the origin the equation becomes
$2X^2 - Y^2 + 3Z^2 + 8XY - 2YZ = 9/2$.

Example 10: Identify the shape of the quadric surface

$$2x^2 - y^2 - z^2 - 2xy - 2xz - 2yz = 1.$$

Solution: Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $Q = \begin{pmatrix} 4 & -1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix}$.

Then the equation has the form $\mathbf{v}^T Q \mathbf{v} = 1$.
$\text{tr}(Q) = 6$, $\text{tr}_2(Q) = 6$ and $|Q| = -4$ and so

$$\chi_Q(\lambda) = \lambda^3 - 6\lambda^2 + 6\lambda + 4$$
$$= (\lambda - 2)(\lambda^2 - 4\lambda - 2)$$

So the eigenvalues of Q are 2, $2 \pm \sqrt{6}$.
After a suitable rotation (we're not asked to describe this rotation) the equation becomes

$$2X^2 + (2 + \sqrt{6})Y^2 + (2 - \sqrt{6})Z^2 = 1.$$

Since $\sqrt{6} > 2$ this has two positive and one negative coefficient and so it is a hyperboloid of one sheet.

Note that to merely identify the shape it isn't necessary to actually find the eigenvalues. Once we've found the characteristic polynomial we only need to determine how many zeros are positive, negative or zero. This can be done by sketching the polynomial.

Example 11: Identify the shape of the quadric surface:

$$x^2 + 2y^2 + 3z^2 + 8xy + 10xz + 12yz = 1.$$

Solution: Let $\mathbf{v} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and $A = \begin{pmatrix} 1 & 4 & 5 \\ 4 & 2 & 6 \\ 5 & 6 & 3 \end{pmatrix}$. The equation can be expressed as $\mathbf{v}^T A \mathbf{v} = 1$.

$\mathrm{tr}(A) = 6$; $\mathrm{tr}_2(A) = -66$; $|A| = 112$, so

$\chi_A(\lambda) = \lambda^3 - 6\lambda^2 - 66\lambda + 112$.

Sketching this polynomial we can see that it has one negative and two positive zeros. Hence the surface is a hyperboloid with one sheet.

EXERCISES FOR CHAPTER 5

Exercise 1: Translate the conic:

$$x^2 + 2y^2 - 3xy + 7x - 10y = 0$$

so that the equation has the form $aX^2 + bY^2 + 2cXY = d$.

Exercise 2: Find a suitable rotation so that the conic: $11x^2 + 9y^2 - 2\sqrt{3}xy = 12$ has the form $aX^2 + bY^2 = c$. Show that the conic is an ellipse and find the lengths of the major and minor axes.

Exercise 3: Consider the conics of the form:

$$5x^2 + 2y^2 + 2kxy = 1$$

for various values of k. Show that, depending on k, this can represent an ellipse, an hyperbola or a pair of straight lines. Find the range of values of k for each of these shapes.

Exercise 4: Identify the quadric surface:

$$z^2 = 3x^2 + 5y^2 + 4xy + 8yz = 1.$$

Exercise 5: Identify the quadric surface:

$$z^2 = 3x^2 + 5y^2 + 4xy + 8yz = 0.$$

SOLUTIONS FOR CHAPTER 5

Exercise 1: Let $X = x + p$, $Y = y + q$.

Then $(X - p)^2 + 2(Y - q)^2 - 3(X - p)(Y - q) + 7(X - p)$
$$- 10(Y - q) = 0.$$

$\therefore\ X^2 - 2pX + p^2 + 2Y^2 - 4qY + 2q^2 - 3XY + 3qX + 3pY$
$$- 3pq + 7X - 7p - 10Y + 10q = 0$$

$\therefore\ X^2 + 2Y^2 - 3XY + (-2p + 3q + 7)X + (-4q + 3p - 10)Y$
$$+ (p^2 + 2q^2 - 3pq - 7p + 10q) = 0$$

We choose p, q so that

$$-2p + 3q = -7 \text{ and}$$
$$3p - 4q = 10.$$

$$\left(\begin{array}{cc|c} -2 & 3 & -7 \\ 3 & -4 & 10 \end{array}\right) \to \left(\begin{array}{cc|c} -2 & 3 & -7 \\ 1 & -1 & 3 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & -1 & 3 \\ 0 & 1 & -1 \end{array}\right).$$

So $q = -1$, $p = 2$.

The translation $X = x + 2$, $Y = y - 1$ translates the conic so that its equation has the form

$$aX^2 + bY^2 + 2cXY = d$$

where $a = 1$, $b = 2$, $c = -3/2$,

$$d = -(p^2 + 2q^2 - 3pq - 7p + 10q) = 12.$$

Exercise 2: Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$.

Then the conic is $\mathbf{v}^T A \mathbf{v} = 12$ where $A = \begin{pmatrix} 11 & -\sqrt{3} \\ -\sqrt{3} & 9 \end{pmatrix}$.

tr(A) = 20, |A| = 96 so

$\chi_A(\lambda) = \lambda^2 - 20\lambda + 96 = (\lambda - 8)(\lambda - 12)$.

The eigenvalues of A are thus 8, 12.

$\boldsymbol{\lambda}$ = 8: $A - 8I = \begin{pmatrix} 3 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \to \begin{pmatrix} \sqrt{3} & -1 \\ 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$

is an eigenvector for $\lambda = 8$.

$\boldsymbol{\lambda}$ = 12: $A - 12I = \begin{pmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & -3 \end{pmatrix} \to \begin{pmatrix} 1 & \sqrt{3} \\ 0 & 0 \end{pmatrix}$ so $\begin{pmatrix} -\sqrt{3} \\ 1 \end{pmatrix}$

is an eigenvector for $\lambda = 12$.

As expected, these eigenvectors are orthogonal to one another.

Let S = $\frac{1}{2}\begin{pmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{pmatrix}$. Note that the columns of this matrix are orthonormal so S is an orthogonal matrix. Moreover $|S| = 1$ and so S is a rotation matrix.

In fact S = $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ where $\cos\theta = ½$ and $\sin\theta = ½\sqrt{3}$. Hence S is the matrix of a 60° rotation.

Let D = $\begin{pmatrix} 8 & 0 \\ 0 & 12 \end{pmatrix}$. Then $A = SDS^{-1} = SDS^T$.

Let $\mathbf{v} = S\mathbf{w}$ where $\mathbf{w} = \begin{pmatrix} X \\ Y \end{pmatrix}$.

Then $\mathbf{v}^TA\mathbf{v} = \mathbf{v}^TSDS^T\mathbf{v} = \mathbf{w}^TD\mathbf{w}$.

Hence the equation transforms to $\mathbf{w}^TD\mathbf{w} = 12$,

that is, $8X^2 + 12Y^2 = 12$.

We can write this as $\frac{X^2}{(\sqrt{3/2})^2} + \frac{Y^2}{1^2} = 1$.

So the conic is an ellipse. The length of the major axis is $2\sqrt{\frac{3}{2}} = \sqrt{6}$ and the length of the minor axis is 2. This new ellipse is obtained from rotating the original ellipse through a rotation of 60° about the origin.

Exercise 3: Let $\mathbf{v} = \begin{pmatrix} x \\ y \end{pmatrix}$ and A = $\begin{pmatrix} 5 & k \\ k & 2 \end{pmatrix}$.

Then the equation can be expressed as $\mathbf{v}^T A\mathbf{v} = 1$.
The shape depends on the values of the eigenvalues.
$\chi_A(\lambda) = \lambda^2 - 7\lambda + (10 - k^2)$.

The eigenvalues are $\alpha = \dfrac{7+\sqrt{49-4(10-k^2)}}{2}$

$= \dfrac{7+\sqrt{9+4k^2}}{2}$ and $\beta = \dfrac{7-\sqrt{9+4k^2}}{2}$.

Clearly $\alpha > 0$ and $\beta < \alpha$.
The conic can be rotated so that its equation is:

$$\alpha X^2 + \beta Y^2 = 1.$$

If $\beta > 0$ this is an ellipse.
If $\beta = 0$ it is a pair of straight lines.
If $\beta < 0$ it's a hyperbola.
Now if $\beta = 0$, $\sqrt{9+4k^2} = 7$, so $9 + 4k^2 = 49$ and hence $k^2 = 10$.
If $k^2 < 10$ then $\beta > 0$. In this case the conic is an ellipse.
If $k^2 > 10$ then $\alpha > 0$ and $\beta < 0$. This gives a hyperbola.

range	**β**	**SHAPE**
$-\sqrt{10} < k < \sqrt{10}$	> 0	ellipse
$k = \pm 10$	$= 0$	pair of straight lines
$k < -\sqrt{10}$ or $k > \sqrt{10}$	< 0	hyperbola

Exercise 4: Let **v** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and A = $\begin{pmatrix} -3 & -2 & 0 \\ -2 & -5 & -4 \\ 0 & -4 & 1 \end{pmatrix}$.

Then the equation is $\mathbf{v}^T A \mathbf{v} = 1$.
tr(A) = – 7, tr_2(A) = – 13, |A| = 59 and so
$\chi_A(\lambda) = \lambda^3 + 7\lambda^2 - 13\lambda - 59$.
Sketching this we see that it has one positive zero and two negative zeros. Hence the quadric surface is a hyperboloid with two sheets.

Exercise 5: Let **v** = $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ and A = $\begin{pmatrix} -3 & -2 & 0 \\ -2 & -5 & -4 \\ 0 & -4 & 1 \end{pmatrix}$.

Then the equation is $\mathbf{v}^T A \mathbf{v} = 0$.
As in exercise 4, $\chi_A(\lambda) = \lambda^3 + 7\lambda^2 - 13\lambda - 59$, which has one positive zero and two distinct negative ones.
The equation, after a suitable rotation, will have the form:

$$\frac{X^2}{a^2} + \frac{Y^2}{b^2} = Z^2.$$

The cross-sections by planes parallel to the X-Y plane will be ellipses, except for the X-Y plane itself, where the cross-section is just the origin. The cross sections by planes parallel to the Y-Z and X-Z planes will be hyperbolas, except for the Y-Z and X-Z planes themselves where the cross-section will be a pair of straight lines. This is not an hyperboloid with one sheet because the constant in the equation is zero. It’s best described as an elliptical cone.