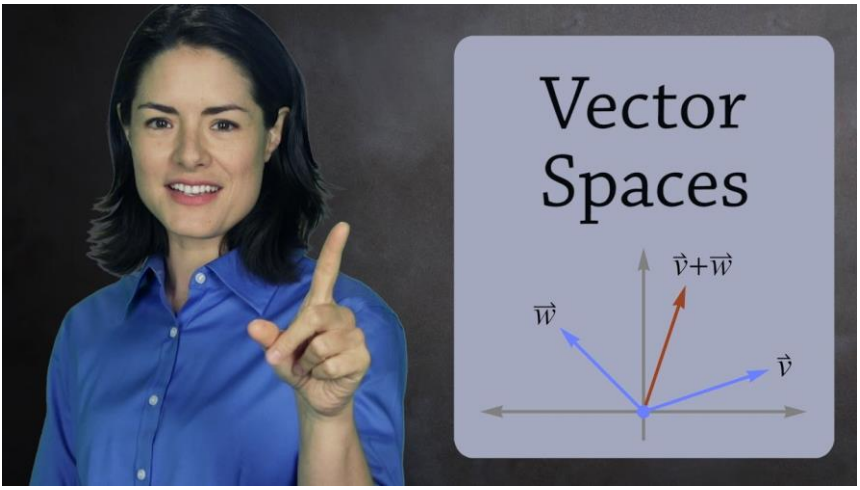
LINEAR ALGEBRA



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These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

INTRODUCTION

Up till now a vector has meant a row or column vector, with several components. But most of the theory applies to "abstract" vectors, mathematical objects that satisfy the Vector Space axioms.

This allows us to use the concepts and techniques of linear algebra in situations where there are no components. The most notable example is the cases of spaces of functions. For example we can add two functions by adding corresponding values. And we can multiply functions by a scalar in the obvious way. What is more, if the functions map a field to itself the axioms of a vector space will hold, and hence all the theorems that can be built up from these axioms.

For example, if $f(x) = \sin^2 x$ and $g(x) = \cos^2 x$, as functions from $\mathbb{R}$ to $\mathbb{R}$, then the well known identity $\sin^2 + \cos^2 x = 1$ can be considered as an equation that states that the sum of these two vectors is the constant function $h(x) = 1$.

In an abstract vector space the functions that take sums to sums and scalar products to scalar product are of great importance. They are called linear transformations and we shall study their properties as well as showing that they are intimately related to matrices.

Inner product spaces are vector spaces with some extra structure that enable us to define lengths and orthogonality. These concepts are useful not just in geometric spaces, where orthogonality is equivalent to

perpendicularity (provided the vectors are non-zero) but orthogonal sets of functions are extremely useful in an area of mathematics called Fourier Theory. Finally, we revist diagonalisation using the power of abstract vector spaces.

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