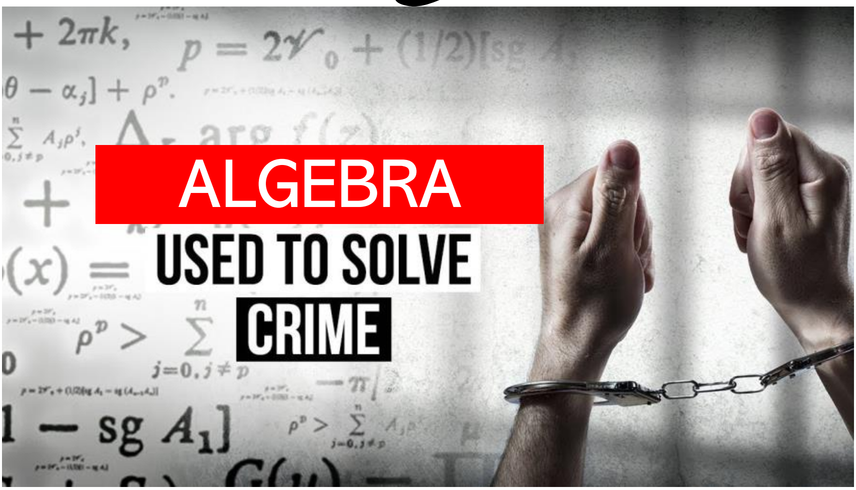
Techniques of Algebra



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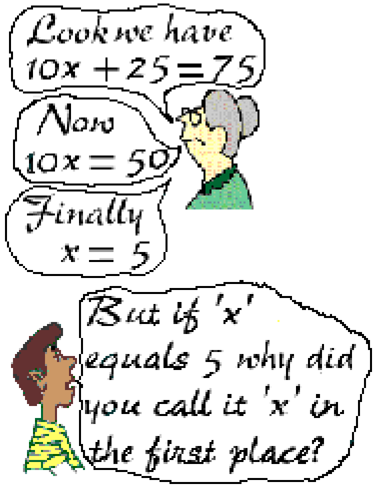
These notes were prepared for students at Macquarie University in Australia but are freely available to anyone. However if you make use of them and are not a Macquarie University student it would be nice if you could email me at christopherdonaldcooper@gmail.com to let me know where you are from. And, if you are from outside of Australia perhaps you could send me a postcard of where you are from to pin up on my wall (Christopher Cooper, 31 Epping Avenue, EASTWOOD, NSW 2122, Australia).

INTRODUCTION

The name “algebra” is derived from the Arabic “al-jabr”, meaning “restoration”. It builds on arithmetic. Instead of carrying out the arithmetic operations of addition, subtraction, multiplication and division with numbers, we include symbols that *represent* numbers.



But algebra goes far beyond the algebra we learnt at school, which is the algebra of real numbers. When we learn about complex numbers we will see that most of high-school algebra works for complex numbers, but not that part of it that involves inequalities.

Polynomials are an important part of algebra and here we take the theory of polynomials somewhat beyond what we learn at school, including solving two polynomial equations in two variables, as well as polynomials in a complex variable.

The final section deals with group theory. This will be your first glimpse of abstract algebra. A group is an algebraic system that consists of a set together with an operation of multiplication. But the elements of the set need not be numbers and the “multiplication” need not have anything to do with multiplication as we know it. All we assume are four basic axioms that regulate the way that this generalized multiplication operates.

While there are groups of numbers, group theory can take us way beyond the confines of mathematics. For example the four ways you can turn a mattress constitute a little group with just four things. A slightly larger group, called the dihedral group of order 8 (meaning that there are 8 things in it) lies behind the kinship rules of the Warlpiri tribe of aborigines (not that they recognised it as a part of group theory).

More important applications are to physics and to certain parts of chemistry. In these notes we merely scratch the surface. We concentrate on finite groups, where we discover that the size of the group gives a lot of information about the structure of the group. There is a set of notes entirely devoted to Group Theory.

An important part of first year mathematics is an area called Linear Algebra, which deals with matrices and abstract vector spaces. These are dealt with in another set of notes.

Most of this material can be found in standard first year algebra textbooks. However there are some novelties, such as the One-Way Euclidean Algorithm in the chapter on integers and divisibility and a discussion of solving systems of polynomial equations in several variables.

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